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Optimization Methods for Parabolized Navier-Stokes Flowfield Reconstruction

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Abstract A parabolized Navier-Stokes (PNS) simulation of a jet engine is used to generate downstream temperature, density, and velocity values in 2-D. The PNS equations are a simplification of the full Navier-Stokes that conflate the time and marching coordinate. Downstream temperature, density and velocity readings are sampled, from which the upfield parameters are to be reconstructed.

We solve this problem by defining a cost functional that describes the lack-of-fit between the model parameters and the upfield initial conditions. Using a continuous adjoint of the PNS equations, the gradient of the cost function with respect to the initial conditions is obtained. From this point, we use the gradient and an initial guess as input to several optimization algorithms, including: three versions of the nonlinear conjugate-gradient (CG) method, three quasi-Newton adaptations, a new hybrid CG/quasi-Newton algorithm, and a non-smooth optimizer n1cv2. Based on our tests for a variety of different Reynolds numbers, the non-smooth n1cv2 method proved superior at reconstructing the initial conditions by a full order of magnitude but at a higher computational cost. Nonsmooth optimization methods for inverse problems of this nature hold promise for increased accuracy.

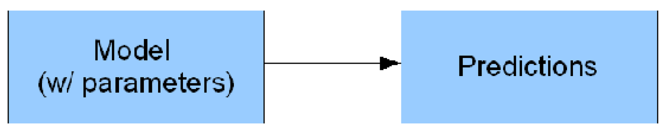
Introduction

Downfield from a supersonic jet, measurements of temperature, velocity, and density are taken. From this data, we wish to reconstruct what conditions at the jet nozzle could have generated this data.

Forward Problem

- **Description**

In a typical forward problem, we have a model with a set of parameters that we use to calculate our data.



Solving a forward problem is usually well-posed, *i.e.* it meets the following three critera:

1. A solution exists
2. The solution is unique
3. The solution depends continuously on the data

In our case, this corresponds to giving conditions at the jet nozzle and solving for the downstream data.

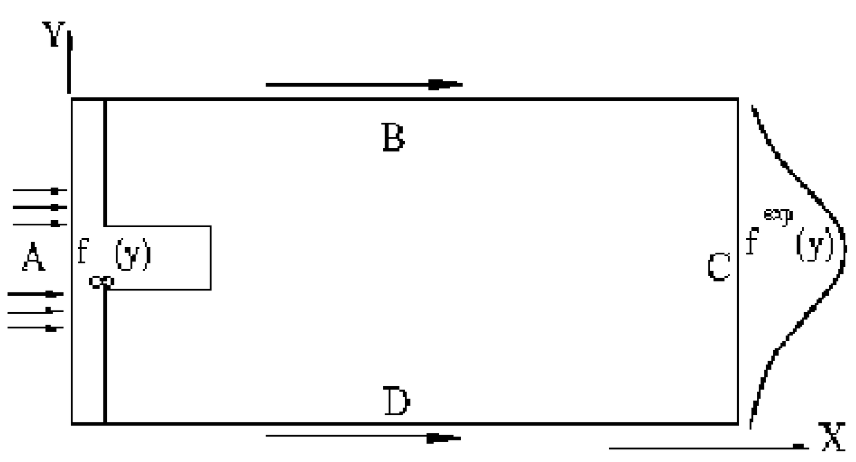
- **Model**

We consider an underexpanded jet in supersonic flow ($M > 1, \rho > \rho_{ambient}$). We use the 2-D non-dimensionalized parabolized Navier-Stokes with a laminar flow assumption (*i.e.* $\|$ layers):

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0$$
$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{1}{\rho \mathrm{Re}}\frac{\partial^2 u}{\partial y^2}$$
$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{4}{3}\frac{1}{\rho \mathrm{Re}}\frac{\partial^2 v}{\partial y^2}$$
$$u\frac{\partial e}{\partial x} + v\frac{\partial e}{\partial y} + (\gamma - 1)e\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = \frac{1}{\rho}\left(\frac{\gamma}{\mathrm{Re}\,\mathrm{Pr}}\frac{\partial^2 e}{\partial y^2} + \frac{4}{3}\frac{1}{\mathrm{Re}}\left(\frac{\partial u}{\partial y}\right)^2\right)$$

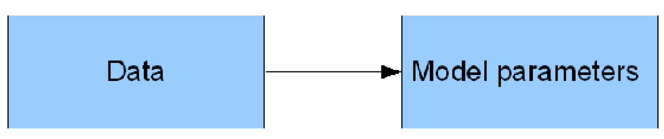
Where:

- u, v are the unknown velocities in the x and y directions,
- ρ is the unknown density, and T is the unknown temperature,
- $p = \rho RT$ is the pressure, where R is the ideal gas constant,
- Re is the Reynolds number and Pr is the Prandtl number,
- $e = C_v T$ is the specific energy, where C_v is the specific volume heat capacity,
- γ is the specific heat ratio,
- $(x, y) \in \Omega$ where Ω is shown below. f_∞ is entrance boundary, and f^{exp} is the outflow boundary.



Inverse problem

In a typical inverse problem, we are given data and want to find a set of parameters for the model that best explain the observations.



Unlike the forward case, an inverse problem is usually ill-posed. Usually the issue is with number 3) from above. In other words, small changes in the data can lead to large variations in the solution parameters.

- **Cost function**

We require a function that describes the lack of fit between the model solution which we will then minimize. We call this function ϵ and it depends upon the initial condition we are trying to solve:

$$\epsilon(f_\infty(y)) = \sum_{m=1}^{M}\int_\Omega (f^{exp}(x,y) - f(x,y))^2 \delta(x - x_m)\delta(y - y_m)\, dxdy$$

Where $f_\infty(y)$ is the initial condition, $f^{exp}(x, y)$ are the known downfield parameters, $f(x, y)$ is the model solution corresponding to f_∞, and x_m, y_m are the grid points of the corresponding measurement.

Numerical Solution

- **Problem Statement:** Given an initial guess $f^0_\infty(y)$, minimize $\epsilon(f_\infty(y))$
- **Adjoint of cost function** Most modern optimization algorithms use gradient information in order to efficiently solve a minimization problem. This gradient can be thought of as the direction of steepest ascent/descent. In our case, we wish to find

$$\nabla\epsilon = \left(\frac{\partial\epsilon}{\partial u_\infty}, \frac{\partial\epsilon}{\partial v_\infty}, \frac{\partial\epsilon}{\partial \rho_\infty}, \frac{\partial\epsilon}{\partial e_\infty}\right)^T$$

In this study we used the differentiate-then-discretize approach to finding our adjoint. The derivations and equations can be found in [1].

- **Optimization Algorithms**

Now that we have our problem formulation and the gradient, we can use a variety of optimization algorithms to compare the results. In this study, we compared:

- Three versions of the non-linear conjugate gradient (CG) method
- The quasi-Newton Broyden-Fletcher-Goldfarb-Shanno (BFGS) methods
- The limited-memory quasi-Newton method (L-BFGS)
- The truncated-Newton method (T-N)
- A hybrid CG and quasi-Newton method
- The INRIA/MODULOPT nonsmooth variable metric bundle algorithm n1cv2[2]

Numerical Results

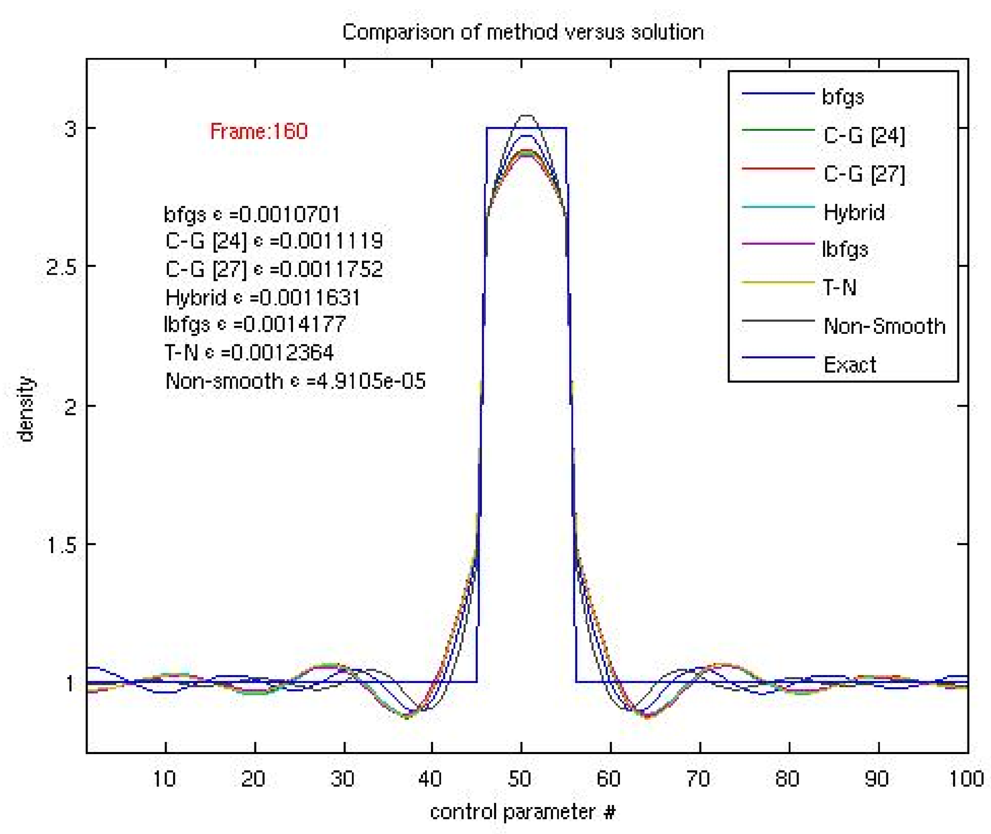


Figure 1: Comparison of the reconstructed solution found by each optimization method.

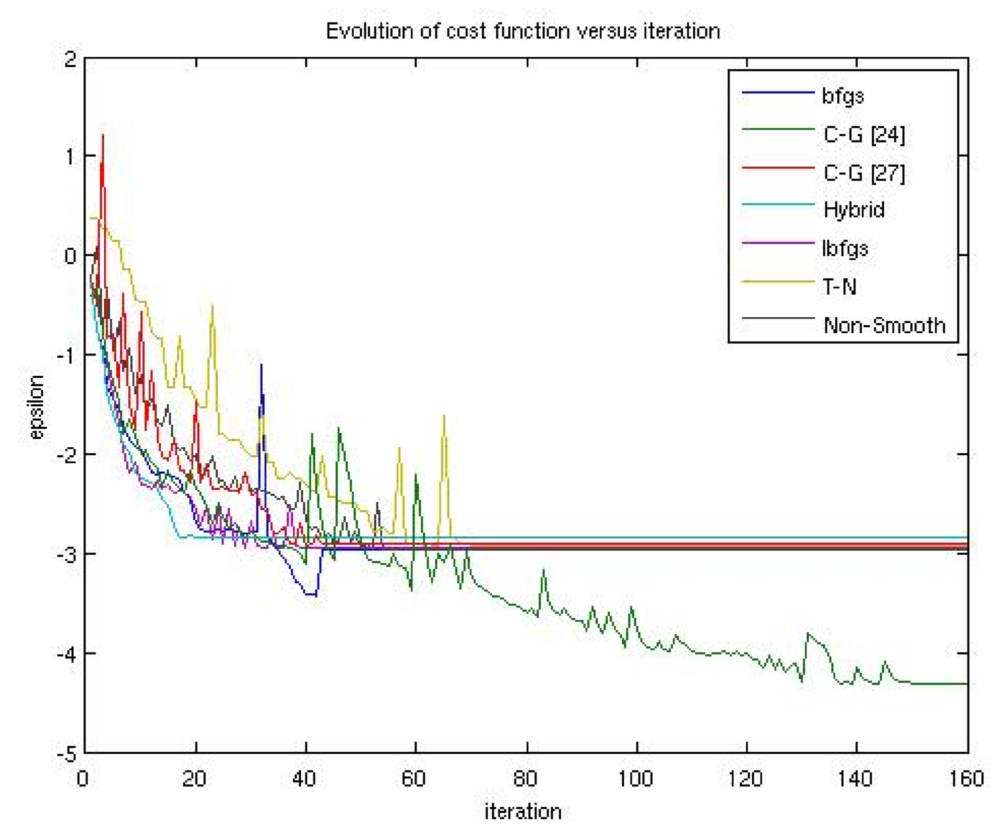


Figure 2: Iteration progress for investigated methods versus the cost function epsilon.

- **Conclusions:**

While all the methods tested were able to find an adequate initial condition, the non-smooth n1cv2 method proved superior at reconstructing the initial conditions by a full order of magnitude.

Nonsmooth optimization methods for inverse problems of this nature hold promise for increased accuracy.

References

[1] Aleksey K. Alekseev, I. M. Navon and J.L. Steward, "Comparison of advanced large-scale minimization algorithms for the solution of inverse ill-posed problems", Optimization Methods and Software, **Vol. 24**, issue 1, pp. 63–87 (Feb. 2009).

[2] C. Lemarechal, C. Sagastizabal, "Variable metric bundle methods: from conceptual to implementable forms", Mathematical Programming, **Vol. 76**, issue 3, pp. 393–410 (Mar. 1997).