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Finite Element Methods for a Peridynamic Model of Mechanics

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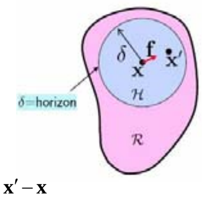
Abstract

Peridynamic is a recently developed theory of solid mechanics that replaces the partial differential equations (PDE) of the classical continuum theory with integro-differential equations (IDE). We apply Finite Element Methods (FEM) to implement the peridynamic model. Since the integro-differential equations remain valid in the presence of discontinuities such as cracks, the method has the potential to model fracture and damage with great generality. We use piecewise constant functions in regions where discontinuities may appear and piecewise linear function in areas where the solutions is smooth and investigate how to combine these two methods. We are also interested in the choice of the horizon radius to implement the peridynamic model more accurately. Theoretical analysis and numerical results for different cases are given.

The Peridynamic Model

The acceleration of any particle at $\mathbf{x}$ in the reference configuration at time t is described by the Integro-Differential Equation (IDE) [1,2]

$$\rho\ddot{\mathbf{u}}(\mathbf{x},t)=\int_{H_{\mathbf{x}}}\mathbf{f}(\mathbf{u}(\mathbf{x}',t)-\mathbf{u}(\mathbf{x},t),\mathbf{x}'-\mathbf{x})\mathrm{d}V_{\mathbf{x}'}+\mathbf{b}(\mathbf{x},t)$$



where ρ is mass density, $\mathbf{u}$ is displacement, $H_{\mathbf{x}}$ is a neighborhood of $\mathbf{x}$ (i.e. a spherical region of radius δ around $\mathbf{x}$, where δ is called the horizon), $\mathbf{f}$ is a pairwise force function (force/volume^2), $\mathbf{b}$ is body force density.

For simplification, we denote the relative position of two particles by $\boldsymbol{\xi}$: $\boldsymbol{\xi}=\mathbf{x}'-\mathbf{x}$

and denote the relative displacement of two particles by $\boldsymbol{\eta}$: $\boldsymbol{\eta}=\mathbf{u}(\mathbf{x}',t)-\mathbf{u}(\mathbf{x},t)$

It is convenient to assume that for a given material such that beyond the horizon δ, the particles do not interact, i.e. $\quad |\xi|>\delta \;\Rightarrow\; \mathbf{f}(\boldsymbol{\eta},\xi)=0,\ \forall\boldsymbol{\eta}.$

Peridynamic Theory	**V.S.**	**Classical Theory of Continuum Mechanics**
Integro-Differential Equation		Partial Differential Equation
⇓		⇓
Do Not Require Partial Derivatives to $\mathbf{x}$		Require Partial Derivatives to $\mathbf{x}$
⇓		⇓
Directly Applied on crack surfaces and other singularities		Need the special techniques of fracture mechanics

1-D Linear PD Model for a Microelastic Material

A linearized version of the Peridynamic theory for a microelastic material takes the form $\mathbf{f}(\boldsymbol{\eta},\xi)=\mathbf{C}(\xi)\boldsymbol{\eta},\ \forall\xi,\boldsymbol{\eta}.$ where $\mathbf{C}(\xi)$ is the material's micromodulus function[2], for the special case of proportional material, that

$$\mathbf{C}(\xi)=c\frac{\xi\otimes\xi}{|\xi|^3}\qquad \text{i.e.}\qquad C_{ij}(\xi)=c\frac{\xi_i\xi_j}{(\xi_k\xi_k)^{3/2}}$$

The constant of proportionality c depends not only on the radius of the peridynamics horizon but also on the the dimension of the domain, and in one-dimensional case, $c=\frac{18k}{5\delta^2}$ [3], where k denotes the bulk modulus. Let $\Omega=(\alpha,\beta),\ \Omega'=(\alpha-\delta,\beta+\delta)$. So a linearized version of the Peridynamic theory for the microelastic material simplifies to the Integro-Differential Equation:

$$\rho\frac{\partial^2u(x,t)}{\partial t^2}=\int_{x-\delta}^{x+\delta}\frac{18k}{5\delta^2}\frac{u(x',t)-u(x,t)}{|x'-x|}\mathrm{d}x'+b(x,t)$$

To simplify the computation, let $\rho=1,\ \ k=\frac{5}{18}$ and including the boundary and initial conditions to the equations, then we get the equations we will concentrate on:

$$\begin{cases}\dfrac{\partial^2u(x,t)}{\partial t^2}=\dfrac{1}{\delta^2}\displaystyle\int_{x-\delta}^{x+\delta}\frac{u(x',t)-u(x,t)}{|x'-x|}\mathrm{d}x'+b(x,t), & x'\in\Omega',x\in\Omega\\ u(x,0)=u_0(x),\quad u_t(x,0)=u_1(x), & x\in\Omega\\ u(x,t)=g(x,t), & x\in\overline{\Omega'}\setminus\Omega,t\in[0,T]\end{cases}$$

FEM Framework for 1-D PD Model

Denote $V=\{u(x,t)\,|\,D_t^{\gamma}u(x,t)\in L^2(\Omega'),\gamma\le 2\}$,
$S=\{u(x,t)\,|\,u(x,t)\in V,u(x,t)=g(x,t)\ \text{ on }\ \overline{\Omega'}\setminus\Omega\}$

Let S_h denote the finite dimensional subspace of S, the Galerkin finite element approximation to the equations is defined as follows: to solve $u_h(x,t)\in S_h$, such that

$$\begin{cases}(u_{htt}(x,t),v_h(x))=\dfrac{1}{\delta^2}(\displaystyle\int_{x-\delta}^{x+\delta}\frac{u_h(x',t)-u_h(x,t)}{|x'-x|}\mathrm{d}x',v_h(x))+(b(x,t),v_h(x)), & \forall v_h(x)\in L^2(\Omega)\\ u_h(x,0)=u_0(x),\quad u_{ht}(x,0)=u_1(x), & x\in\Omega\end{cases}$$

Let $u_h(x,t)=\sum_i U_{h,i}(t)\phi_i(x)$, $v_h(x)=\phi_j(x)$ then

$$\sum_i(\phi_i(x),\phi_j(x))\frac{d^2}{dt^2}U_{h,i}(t)=\frac{1}{\delta^2}(\int_{x-\delta}^{x+\delta}\frac{\sum_i U_{h,i}(t)(\phi_i(x')-\phi_i(x))}{|x'-x|},\phi_j(x))+(b(x,t),\phi_j(x))$$

The resulting system is a linear system, denote $A=\{A_{ji}\}_{N\times N}$, where $A_{ji}=\int_\alpha^\beta\phi_j(x)\int_{x-\delta}^{x+\delta}\frac{\phi_i(x)-\phi_i(x)}{|x'-x|}\mathrm{d}x'\mathrm{d}x$ and if we let $\delta=\delta_0=K\Delta x$, then for example if we use the piecewise constant function as the basis function to solve the system, A is a $2K+1$ banded positive definite matrix.

References

[1] S. A. Silling, Reformulation of Elasticity Theory for Discontinuities and Long-Range Forces. *J. Mech. Phys. Solids* 48 (2000). pp. 175-209.

[2] S. A. Silling and E. Askari, A Meshfree Method Based on the Peridynamic Model of Solid Mechanics, Computers and Structures, Vol. 83 (2005) 1526-1535.

[3] E. Emmrich and O. Weckner: The peridynamic equation of motion in non-local elasticity theory. III European Conference on Computational Mechanics Solids, Structures and Coupled Problems in Engineering (Lisbon, June 2006), Springer, 2006, 19 p.

Numerical Results

Notice that if we use the Taylor's Theorem to the Integral Term in the equation, we get that

$$\frac{1}{\delta^2}\int_{x-\delta}^{x+\delta}\frac{u(x')-u(x)}{|x'-x|}\mathrm{d}x'=\frac{1}{2}u''(x)+\frac{1}{48}u^{(4)}(x)\delta^2+H.O.T$$

Which means that if u is smooth enough, $\lim_{\delta\to 0}\frac{1}{\delta^2}\int_{x-\delta}^{x+\delta}\frac{u(x')-u(x)}{|x'-x|}\mathrm{d}x'=\frac{1}{2}u''(x)$

We first exam the convergence rate for the case when exact solution $u(x)$ are polynomials, whose degrees are less than 4: for example, $u(x)=x^2$, we get the error for two cases: (1) δ is not fixed ; (2).δ is fixed as below:

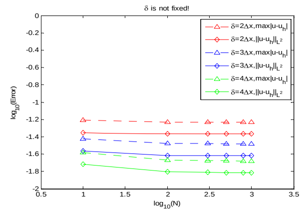


Figure 1

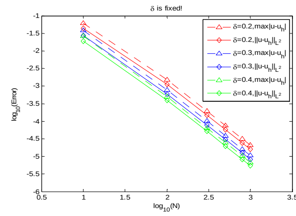


Figure 2

From Figure 1, the δ is not fixed, we can see that: as the number of grid point increases, the error does not decrease, almost fixed, which means that the numerical solution is not convergent to the exact solution as we want! And also from the result we can see that when δ is larger, the error is smaller!

From Figure 2, the δ is fixed, we can see that as the number of grid point increases, the error decreases as we want, which means that the numerical solution is convergent to the exact solution, for the case when exact solution $u(x)=x$ we can get the error to be 10^{-16} and also from the result we can see that when δ is larger, the error is smaller!

Since the equations do not require derivative to $\mathbf{x}$, it can deal with the case when the solution is discontinuous :

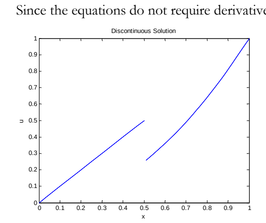


Figure 3

Manufactured discontinuous solution:

$$u(x)=\begin{cases}x\ ,x\in[0.0,0.5)\\ x^2,x\in[0.5,1.0]\end{cases}$$

	$\delta=0.2$		$\delta=0.3$		$\delta=0.4$	
Δx	Error(L_2)	Rate	Error(L_2)	Rate	Error(L_2)	Rate
1/9	0.0227	/	0.0142	/	0.0102	/
1/27	0.0047	1.4334	0.0026	1.5453	0.0018	1.5789
1/81	7.1006E-04	1.7203	3.8508E-04	1.7384	2.5751E-04	1.7699
1/243	9.6250E-05	1.8190	5.1502E-05	1.8312	3.4160E-05	1.8387
1/729	1.2483E-05	1.8592	6.6258E-06	1.8666	4.3720E-06	1.8713
1/2187	1.5800E-06	1.8814	8.3379E-07	1.8867	5.4809E-07	1.8901

Table 1

From Table 1, we can see that the model works for the discontinuous case, and as the number of grid point increases, the error decreases as we want, which means that the numerical solution is convergent to the exact solution, and also from the result we can see that when δ is larger, the error is smaller; the convergence rate is between [1.5, 2.0].

Theoretical Analysis

We try to show the **Existence and Uniqueness** of the finite element solution to 1-D linearized peridynamic model by Lax-Milgram theorem:

We Ignore the u_{tt} term first, and without loss of generality let the boundary conditions to be $u(x)=0$ on $\overline{\Omega'}\setminus\Omega$ then we can simplify the finite element equation to be:

$$-\int_\alpha^\beta\int_{x-\delta}^{x+\delta}\frac{u(x')-u(x)}{|x'-x|}\mathrm{d}x'v(x)\mathrm{d}x=\delta^2\int_\alpha^\beta b(x)v(x)\mathrm{d}x$$

$$A(u,v)=F(v)$$

$$A(u,v)=\int_\alpha^{\alpha+\delta}\int_{x-\delta}^{\alpha}\frac{u(x)v(x)}{x-x'}\mathrm{d}x'\mathrm{d}x-\int_\alpha^\beta\int_{\max(\alpha,x-\delta)}^{\min(\beta,x+\delta)}\frac{u(x')-u(x)}{|x'-x|}\mathrm{d}x'v(x)\mathrm{d}x+\int_{\beta-\delta}^{\beta}\int_\beta^{x+\delta}\frac{u(x)v(x)}{x'-x}\mathrm{d}x'\mathrm{d}x$$

Define the new Norm by $|||u|||=A(u,u)^{1/2}$, in order to use the Lax-Milgram theorem, we need to prove the equivalent between two norms: i.e. $C_1\|u\|_0^2\le|||u|||^2\le C_2\|u\|_1^2$.

First, by the symmetric property, we can prove that:

$$-\int_\alpha^\beta\int_{\max(\alpha,x-\delta)}^{\min(\beta,x+\delta)}\frac{u(x')-u(x)}{|x'-x|}\mathrm{d}x'v(x)\mathrm{d}x=\frac{1}{2}\int_\alpha^\beta\int_{\max(\alpha,x-\delta)}^{\min(\beta,x+\delta)}\frac{u(x')-u(x)}{|x'-x|}(v(x')-v(x))\mathrm{d}x'\mathrm{d}x$$

Then by the "Morrey's inequality ", we can prove that:

$$\frac{1}{2}\int_\alpha^\beta\int_{\max(\alpha,x-\delta)}^{\min(\beta,x+\delta)}\frac{u(x')-u(x)}{|x'-x|}(v(x')-v(x))\mathrm{d}x'\mathrm{d}x\le C_2\|u\|_1^2$$

So the only step we need to prove is the left side of the inequality, which is in progress.

Summary and Future works

We have successfully applied the Finite Element Methods to implement the 1-D linearized peridynamics model, we get the convergent numerical results; compare the error for cases when δ is fixed or not; exam the influence of the size of the horizon to the solution and finish some part of the theoretical analysis of the existence and uniqueness of the finite element solution to 1-D linearized peridynamic model.

In the future, we will use piecewise constant functions in regions where discontinuities may appear and piecewise linear function in areas where the solutions is smooth and investigate how to combine these two methods. We will also continue the theoretical analysis and numerical results for different cases.