Numerical Methods for Backward Stochastic Differential Equations in High Dimensions

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Abstract

We consider the numerical solutions of the following backward stochastic differential equation(BSDE):

$$\begin{cases} -dy_t = f(t, y_t, z_t)dt - z_t dW_t, & t \in [0, T), \\ y_T = \xi, \end{cases} \tag{1}$$

which are of great importance in financial mathematics, stochastic control, stock markets, turbulent fluid flow ,etc. We propose a stable multi-step scheme on time-space grids for solving BSDEs. The integrands are approximated, in the time direction, by using Lagrange interpolating polynomials with values at multi-time levels. In the spatial directions, we construct a new kind of stochastic process, called Gauss-Hermite process, by which the integrands are approximated accurately and the spatial complexity can be reduced significantly. For high-dimensional problems, the sparse grids are employed to interpolate the solutions in the spatial directions, such that the high-dimensional BSDEs can be solved much more efficiently with relatively high accuracy.

The Discrete Scheme

For simplicity, we consider the one-dimensional case on a uniform grid of the time interval $[0, T]$ with N steps. Assume $\Delta W_s = W_s - W_{t_n}$ for $s \geq t_n$, is a standard Brownian motion following $N(0, s - t_n)$, and $(y_t, z_t) : [0, T] \times \Omega \to \mathbb{R} \times \mathbb{R}$ is the exact solution of the BSDE (1). For a given positive integer k satisfying $1 \leq k \leq N$, it is easy to obtain

$$y_{t_n} = y_{t_{n+k}} + \int_{t_n}^{t_{n+k}} f(s, y_s, z_s)\, ds - \int_{t_n}^{t_{n+k}} z_s\, dW_s. \tag{2}$$

- **The reference equations**

  Multiply both sides of the equation (2) by $\Delta W_{t_{n+l}}$ where l is a positive integer and $1 \leq l \leq N$, then take the conditional mathematical expectation $\mathbb{E}_{t_n}^x[\cdot]$ to both sides of the derived equation and the equation (2), two reference equations for y_t and z_t are obtained respectively, i.e.

$$\begin{aligned} y_{t_n} &= \mathbb{E}_{t_n}^x[y_{t_{n+k}}] + \int_{t_n}^{t_{n+k}} \mathbb{E}_{t_n}^x[f(s, y_s, z_s)]ds \\ 0 &= \mathbb{E}_{t_n}^x[y_{t_{n+l}} \Delta W_{t_{n+l}}] + \int_{t_n}^{t_{n+l}} \mathbb{E}_{t_n}^x[f(s, y_s, z_s)\Delta W_s]ds - \int_{t_n}^{t_{n+l}} \mathbb{E}_{t_n}^x[z_s]ds. \end{aligned} \tag{3}$$

- **The discretization scheme**

  All the conditional mathematical expectations are approximated by using Lagrange interpolating polynomials with their values at K_y and K_z levels in the time direction; and approximated by Gauss-Hermite quadrature rule in the spatial directions. So that we obtain the following discretization scheme

$$\begin{aligned} y_i^n &= \hat{\mathbb{E}}_{t_n}^{x_i}[\hat{y}^{n+k}] + k\Delta t \sum_{j=1}^{K_y} b_{K_y,j}^k \hat{\mathbb{E}}_{t_n}^{x_i}[f(t_{n+j}, \hat{y}^{n+j}, \hat{z}^{n+j})] \\ &\quad + k\Delta t f(t_n, y_i^n, z_i^n), \\ 0 &= \hat{\mathbb{E}}_{t_n}^{x_i}[\hat{z}^{n+l}] + \sum_{j=1}^{K_z} b_{K_z,j}^l \hat{\mathbb{E}}_{t_n}^{x_i}[f(t_{n+j}, \hat{y}^{n+j}, \hat{z}^{n+j})\Delta W_{t_{n+j}}] \\ &\quad - \sum_{j=1}^{K_z} b_{K_z,j}^l \hat{\mathbb{E}}_{t_n}^{x_i}[\hat{z}^{n+j}] - b_{K_z,0}^l z_i^n. \end{aligned}$$

An Efficient Scheme by Gauss-Hermite Process

For the approximation of the conditional mathematical expectations in space, Gauss-Hermite quadrature rule results in expensive computations, so that we construct the Gauss-Hermite process, combining the Gauss-Hermite quadrature rule and the properties of Brownian motion, to get the more efficient scheme.

- Disadvantages of the Gauss-Hermite quadrature
  - Losing the random properties
  - Having expensive cost in space
- Advantages of the Gauss-Hermite process
  - Considering the random properties
  - Having much less cost in space
  - Keeping the same accuracy as before

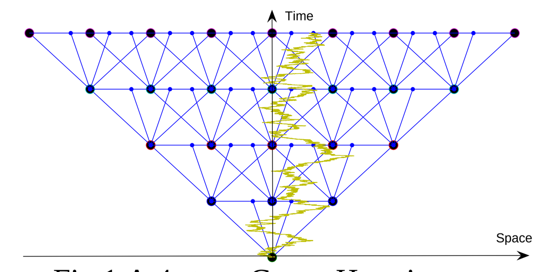


Fig.1 A 4 steps Gauss-Hermite process

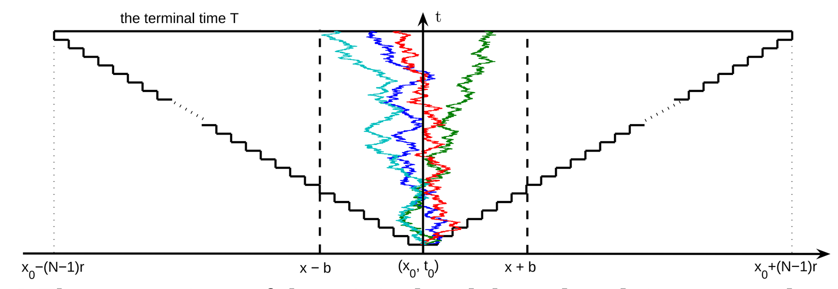
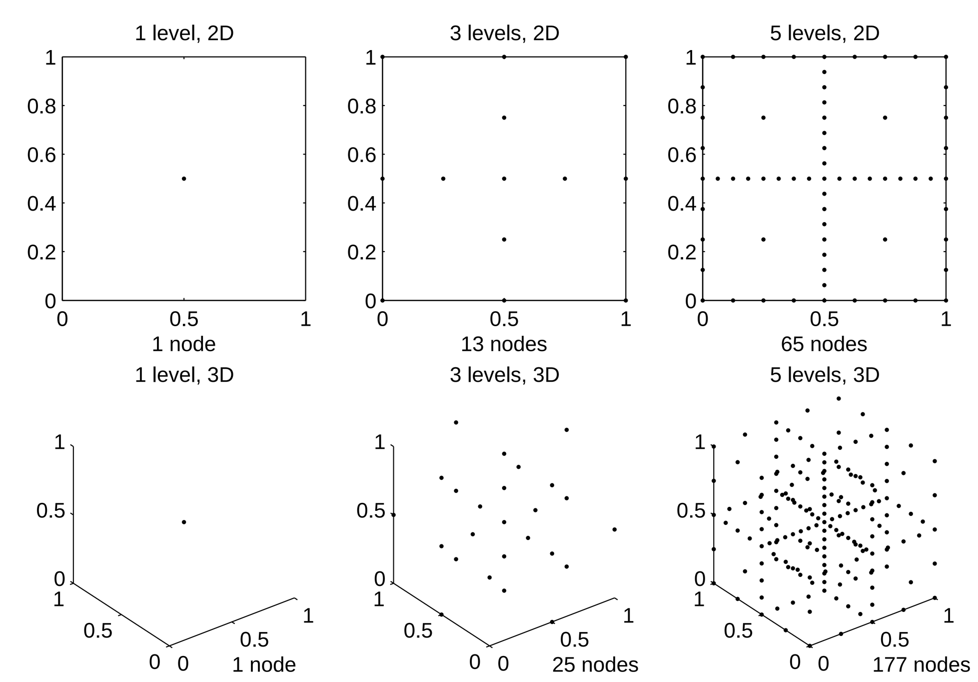
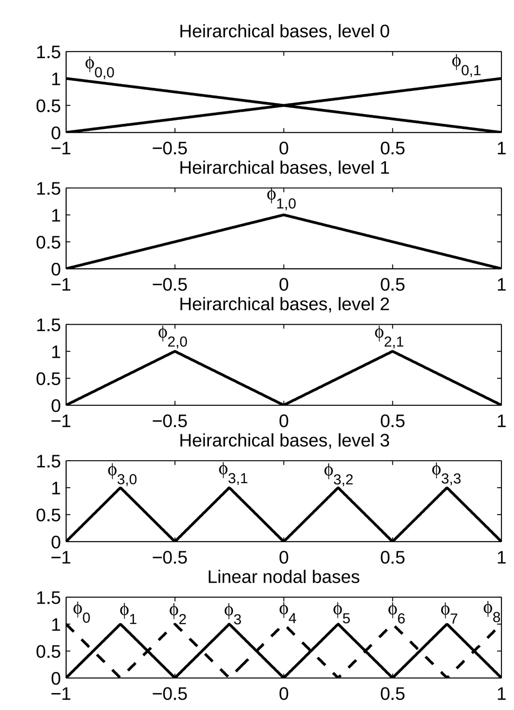


Fig.2 The comparison of the original and the reduced time-space domain

Sparse Grids

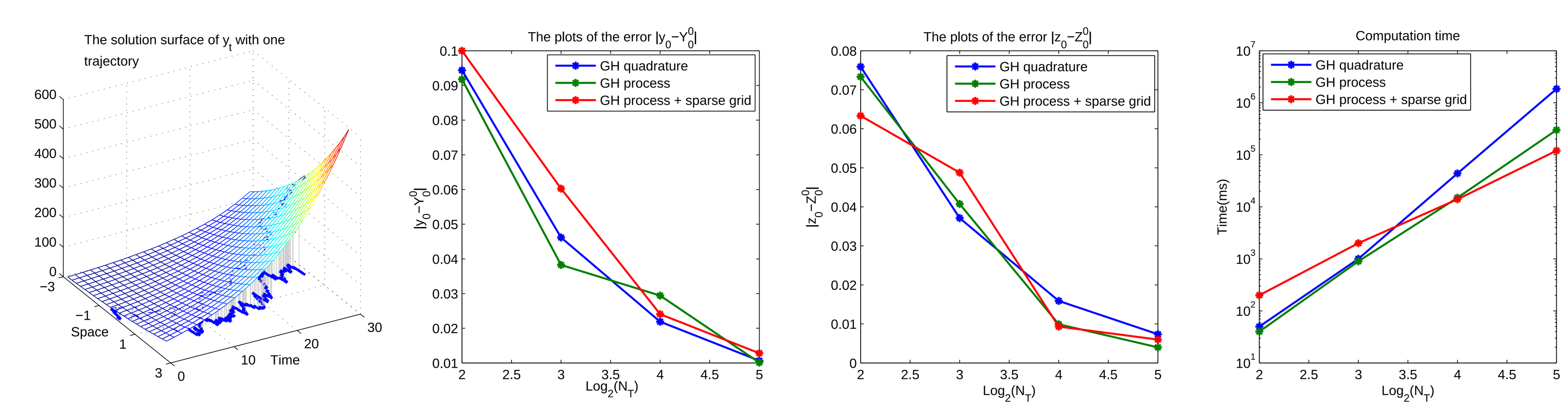
For a d-dimensional BSDE, the discretization on the full spatial grids involves $O(N^d)$ degrees of freedom, where N is the number of grid points in one coordinate direction. Obviously, the complexity will grow up exponentially. Therefore, we construct the sparse grids to discretize the spatial domain such that only $O(N \cdot (\log N)^{d-1})$ degrees of freedom are involved. Then the solutions y_t and z_t can be interpolated at all needed Gauss points by the associated hierarchical bases instead of nodal bases in space. In this case, high dimensional problems can be handled efficiently. The results of numerical experiments will be given in the next section.





Numerical Experiments

Here, we solve the 2 dimensional Black-Scholes equation to show the efficiency and accuracy of our scheme. The results show that the scheme by using Gauss-Hermite process model is so efficient that the computational time decreases by almost one order of magnitude. For the effectiveness of the sparse grids, it becomes even worse for cases with small time steps due to the cost of the construction and related operations of the sparse grids. However, the advantages of the sparse grids show up when dealing with the problem on a time-space grid of large size.



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