Analysis and Implementation of Stochastic Collocation Method for Parabolic Partial Differential Equations with Random Input Data

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Introduction

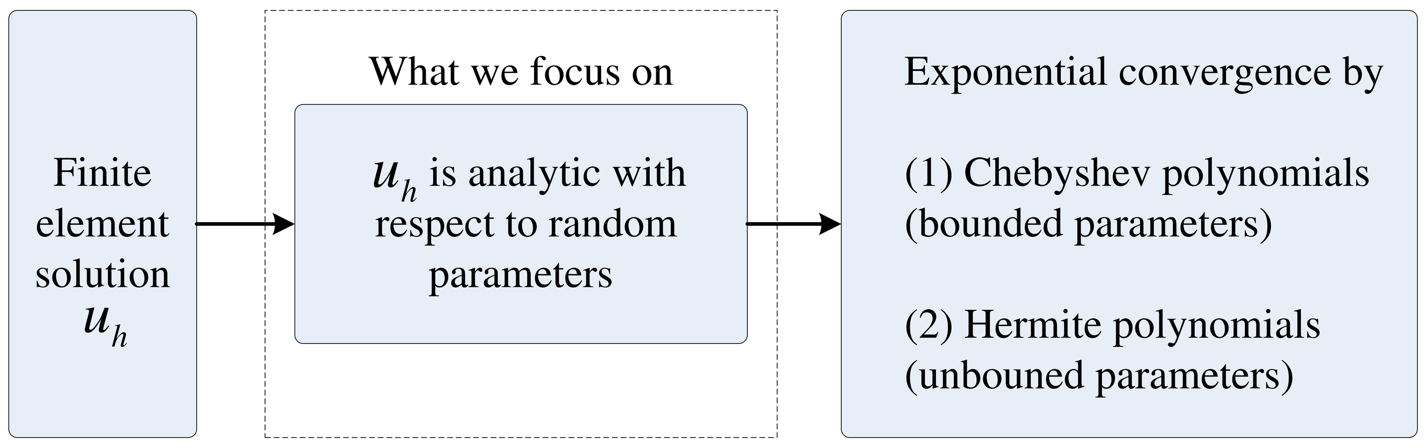
We consider the following linear parabolic stochastic PDE,

$$\begin{aligned} \partial_t u(t,x,\omega) - \nabla\cdot[a(x,\omega)\nabla u(t,x,\omega)] &= f(t,x,\omega) \quad \text{in} \quad [0,T]\times D\times\Omega,\\ u(t,x,\omega) &= 0 \quad \text{on} \quad [0,T]\times\partial D\times\Omega \\ u(0,x,\omega) &= u_0 \quad \text{on} \quad D\times\Omega, \end{aligned} \tag{1}$$

where $D\in\mathbb{R}^d$, Ω is the sample space and $u:\Omega\times[0,T]\times\overline{D}\to\mathbb{R}$. The symbol ∇ means differentiation with respect to the spatial variable $x\in D$. Unlike previous literatures on this issue, we consider a wider range of situations as follows:

- The coefficient and the forcing term are represented not only by KL expansion but also by a nonlinear function of a random vector;
- Both bounded and unbounded random variables are considered;
- Errors are analyzed for both a semi-discrete scheme and a fully-discrete scheme for the parabolic PDE (1).

The error of the numerical solution splits into $\varepsilon = (u-u_h)+(u_h-u_{h,p})$. We mainly analyze the second term, i.e. the interpolation error in the probability space and obtain the first term by classic finite element analysis. Moreover, as long as u_h is analytic with respect to the random parameters, the interpolation error will decay exponentially by using typical approximation theories. Therefore, in what follows the analyticity of u_h is the key point of our analysis. The procedure of the error analysis is shown in the flowchart below:



The Stochastic Collocation Method

For a fixed $T>0$, the weak formulation of (1) has the following three equivalent forms:

$$\int_D \mathbb{E}[\partial_t uv]dx + \int_D \mathbb{E}[a\nabla u\cdot\nabla v]dx = \int_D \mathbb{E}[fv]dx \quad \forall v\in H_0^1(D)\otimes L_P^2(\Omega)$$

$$\Updownarrow KL-expansion$$

$$\int_\Gamma\int_D \partial_t uv\rho dy + \int_\Gamma\int_D [a\nabla u\cdot\nabla v]\rho dy = \int_\Gamma\int_D fv\rho dy \quad \forall v\in H_0^1(D)\otimes L_\rho^2(\Gamma) \tag{2}$$

$$\Updownarrow$$

$$\int_D \partial_t u(y)vdx + \int_D a(y)\nabla u(y)\cdot\nabla vdx = \int_D f(y)vdx \quad \forall v\in H_0^1(D)\otimes L_\rho^2(\Gamma),\ \rho\text{−a.e. in }\Gamma$$

Then, an approximation is constructed with the stochastic collocation method by

- For a fixed T, construct an approximation $u_h(T,\cdot,y):\Gamma\to H_h(D)$ by projecting (2) onto the subspace $H_h(D)$, i.e. for each $y\in\Gamma$

$$\int_D \partial_t u_h(y)v_h dx + \int_D a(y)\nabla u_h(y)\cdot\nabla v_h dx = \int_D f(y)v_h dx \quad \forall v_h\in H_h(D). \tag{3}$$

- Collocating (3) on the zeros of orthogonal polynomials and building the discrete solution $u_{h,p}\in H_h(D)\otimes\mathcal{P}_p(\Gamma)$ by interpolating in y the collocated solutions, i.e.

$$u_{h,p}(T,x,y) = \mathcal{I}_p u_h(T,x,y) = \sum_{j_1=1}^{m_1}\cdots\sum_{j_d=1}^{m_d} u_h(T,x,y_{j_1},\cdots,y_{j_d})(l_{j_1}\otimes\cdots\otimes l_{j_d}). \tag{4}$$

where, for instance, the function $\{l_{j_k}\}_{k=1}^d$ can be taken as Lagrange polynomials.

Error Analysis of the Semi-discrete Scheme

Lemma 1 *For any $T>0$, if the solution $u(T,x,y_n,y_n^*)$ is a function of y_n, $u:\Gamma_n\to C^0_{\sigma_n^*}(\Gamma_n^*;L^2(L$ then the k-th derivative of $u(T,x,y)$ with respect to y_n satisfies*

$$\|\partial_{y_n}^k u(T,y)\|_{L^2(D)} \le Ck!(2\gamma_n^k) \tag{5}$$

where $\gamma_n>0$, C depends on $\|f(y)\|_{L^2(0,T;D)}$, $\|u_0(y)\|_{L^2(D)}$, a_{min} and the Poincaré coefficient C_p.

Theorem 1 *Under Lemma 1, the solution $u(t,x,y_n,y_n^*)$ as a function of y_n admits an analytic extension $u(z,y_n^*)$, $z\in\mathbb{C}$, in the region of the complex plane*

$$\Sigma(\Gamma_n,\tau_n) := \{z\in\mathbb{C}, dist(z,\Gamma_n)\le\tau_n\} \tag{6}$$

with $0<\tau_n<1/(2\gamma_n)$.

Theorem 2 *For a fixed $T>0$, by Theorem 1, there exist positive constants $r_n, n=1,2,\cdots,d$, and C which is independent of h and p, such that*

$$\|u_h(T)-u_{h,p}(T)\|_{L^2(D)\otimes L^2_\rho(\Gamma)} \le C\sum_{n=1}^d \beta_n(p_n)\exp(-r_n p_n^{\theta_n}), \tag{7}$$

where if Γ_n is bounded, $\theta_n=\beta_n=1$ and $r_n=\log\left[\frac{2\tau_n}{|\Gamma_n|}\left(1+\sqrt{1+\frac{|\Gamma_n|^2}{4\tau_n^2}}\right)\right]$, if Γ_n is unbounded, $\theta_n=\frac{1}{2}$, $\beta_n=O(\sqrt{p_n})$ and $r_n=\tau_n\delta_n$. τ_n is smaller than the distance between Γ_n and the nearest singularity in the complex plane, as defined in Theorem 1.

Error Analysis of the Fully-discrete Scheme

The fully-discrete Crank-Nicolson scheme of the problem (1) is

$$\left(\frac{U^m-U^{m-1}}{\Delta t},v\right)+\left(a\frac{\nabla U^m+\nabla U^{m-1}}{2},\nabla v\right)=(f(t_{m-\frac{1}{2}}),v), \quad \forall v\in S_h, m\ge 1 \tag{8}$$

where S_h is the finite element space and $U^0=u_{0,h}$.

Lemma 2 *If the solution $U^N(x,y_n,y_n^*)$ is a function of y_n, $U^N:\Gamma_n\to C^0_{\sigma_n^*}(\Gamma_n^*;L^2(D))$, and we define one kind of discrete norm as*

$$M(N,l)=\left[\frac{\Delta t}{2}\sum_{j=1}^N \|\sqrt{a}\partial_{y_n}^l(\nabla U^j+\nabla U^{j-1})\|^2_{L^2(D)}\right]^{\frac{1}{2}}, \tag{9}$$

then the k-th derivative of $U^N(x,y)$ with respect to y_n satisfies

$$\|\partial_{y_n}^k U^N(y)\|_{L^2(D)} \le Ck!(2\gamma_n^k) \tag{10}$$

where $\gamma_n>0$, C depends on $\|f(y)\|_{L^2(0,T;D)}$, $\|u_0(y)\|_{L^2(D)}$ a_{min} and the Poincare coefficient C_p.

Theorem 3 *Under Lemma 2, the fully discrete solution $U^N(x,y_n,y_n^*)$ as a function of y_n admits an analytic extension $U^N(z,y_n^*)$, $z\in\mathbb{C}$, in the region of the complex plane*

$$\Sigma(\Gamma_n,\tau_n) := \{z\in\mathbb{C}, dist(z,\Gamma_n)\le\tau_n\} \tag{11}$$

with $0<\tau_n<1/(2\gamma_n)$.

Theorem 4 *For a positive integer N, consider a uniform partition of $[0,T]$ with $\Delta t=T/N$, by Theorem 3, there exist positive constants $r_n, n=1,2,\cdots,d$, and C which is independent of h and p, such that*

$$\|U^N-U_p^N\|_{L^2(D)\otimes L^2_\rho(\Gamma)} \le C\sum_{n=1}^d \beta_n(p_n)\exp(-r_n p_n^{\theta_n}), \tag{12}$$

where θ_n, β_n and r_n are defined as in Theorem 3.

A Numerical Example

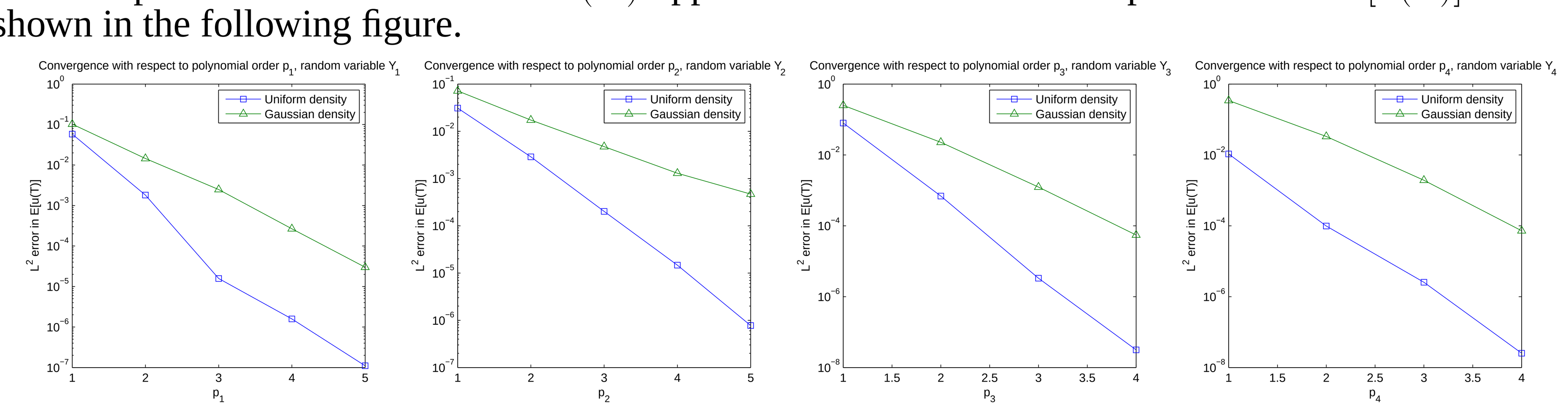
We consider an one-dimensional parabolic PDE:

$$\begin{aligned} \partial_t u - \nabla\cdot(a\nabla u) &= f && \text{on } [0,T]\times D\times\Omega,\\ u(t,a,\omega) &= 0 && \text{on } [0,T]\times\Omega,\\ -a\partial_n u(t,b,\omega) &= 1 && \text{on } [0,T]\times\Omega,\\ u(0,x,\omega) &= 0 && \text{on } D\times\Omega. \end{aligned} \tag{13}$$

where

$$\begin{aligned} a(x,\omega) &= a_{min}+\exp\left[Y_1(\omega)\cos(\pi x)+Y_2(\omega)\sin(\pi x)\right]\\ f(t,x,\omega) &= 100+\exp\left(Y_3(\omega)\cos(\pi x)+Y_4(\omega)\sin(\pi x)\right) \end{aligned} \tag{14}$$

The computational results for the $L^2(D)$ approximation error in the expected value $\mathbb{E}[u(T)]$ are shown in the following figure.



References

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[2] F. Nobile, R. Tempone, *Analysis and implementation issues for the numerical approximation of parabolic equations with random coefficients*, International Journal for Numerical Methods in Engineering, 80(2009), pp.979-1006