Timestepping for a Nonlocal Diffusion Process

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Abstract We consider the time-dependent diffusion equation, formulated in the nonlocal peridynamics setting. Initial and boundary data are given. Time derivatives are approximated using either an explicit Forward Euler, or implicit Backward Euler scheme. The standard timestep stability condition must be reformulated, in light of the peridynamics formulation. We show the results, and the computational implications.

One dimensional nonlocal diffusion problem

Let $Q = \Omega_I \times (0,T]$, where $\Omega_I = (a,b), T > 0$; we define $\Omega_B = [a-\delta, a] \cup [b, b+\delta]$, $\Omega = \Omega_B \cup \Omega_I$,

$$\rho(|x-x'|) = \begin{cases} \frac{1}{|x-x'|^{1+2s}}, & |x-x'| \le \delta, \\ 0, & |x-x'| > \delta. \end{cases}$$

We define

$$V_D = \{v \in L_2(\Omega) : v = 0 \text{ on } \Omega_B\}$$

and let the space associated with the kernel ρ be:

$$S(\Omega) = \left\{ u \in V_D : \int_\Omega \int_\Omega \rho(|x-x'|)(u(x)-u(x'))^2 dx' dx < +\infty \right\}$$

which is the same as the definition in [1].

The equation is:

$$\frac{\partial u}{\partial t} = -c^2 \int_{x-\delta}^{x+\delta} \frac{u(t,x)-u(t,x')}{|x-x'|^{1+2s}} dx' + f(t,x),\ x \in \Omega_I, \quad (1)$$
$$u(x,t) = 0,\ x \in \Omega_B,\ t \in [0,T],$$
$$u(x,0) = u_0(x),\ x \in \Omega_I.$$

As we know, the nonlocal operator here is strictly connected with the fractional laplacian $\bigtriangledown^{2s}$, for $0 < s < 1$. The case $\frac{1}{2} < s < 1$ is treated in [3]. Here we concentrate on the case $0 \le s \le 1/2$.

Lemma and Theorems

For $u, v \in S(\Omega)$, $a(\cdot,\cdot)$ is defined by

$$\begin{aligned} a(u,v) &= \int_\Omega \int_\Omega \rho(|x-x'|)(u(x)-u(x'))v(x) dx' dx \\ &= \frac{1}{2} \int_\Omega \int_\Omega \rho(|x-x'|)(u(x)-u(x'))(v(x)-v(x')) dx' dx \end{aligned} \quad (2)$$

Lemma 1 [1]. The bilinear form given in equation (2), with kernel ρ, has the estimate

$$\lambda \|u\|^2_{L_2(\Omega)} \le a(u,u),\ \forall u \in S(\Omega).$$

We also have the estimate:

$$\lambda = \frac{9\delta}{(2^{2L+5} - 24L - 32)(b-a+2\delta)^{1+2s}},\ L = \left[\frac{b-a}{\delta}\right] + 1.$$

Approximating (1) by the Backward and Forward Euler schemes on a uniform mesh, we have the theorems:

Theorem 1. The Backward Euler scheme for nonlocal diffusion equation (1) is unconditionally stable. If the solution $u(x,t)$ of (1) is sufficiently smooth, then the Backward Euler scheme for the nonlocal diffusion equation (1) converges and we have the estimate:

$$\max_{1 \le m \le M} \|u(\cdot, t^m) - u_h^m\|_{L_2(\Omega)} \le C(h^2 + \Delta t),$$

where C is a positive constant independent of h and Δt.

Theorem 2. The Forward Euler scheme for nonlocal diffusion equation (1) is conditionally stable provided that:

$$\Delta t \le \alpha h,\ \gamma \in (0, c\sqrt{\lambda/2}),\ \alpha = \frac{\lambda c^2 - 2\gamma^2}{3\lambda\beta(1+\gamma)c^4},\ \beta \text{ depends on } s, h.$$

If the solution $u(x,t)$ of (1) is sufficiently smooth, and Δt and h satisfy the stability condition, then the Forward Euler scheme for the nonlocal diffusion equation (1) converges and we have the estimate:

$$\max_{1 \le m \le M} \|u(\cdot, t^m) - u_h^m\|_{L_2(\Omega)} \le C(h^2 + \Delta t),$$

where C is a positive constant independent of h and Δt.

For the numerical experiments, we consider the following example:

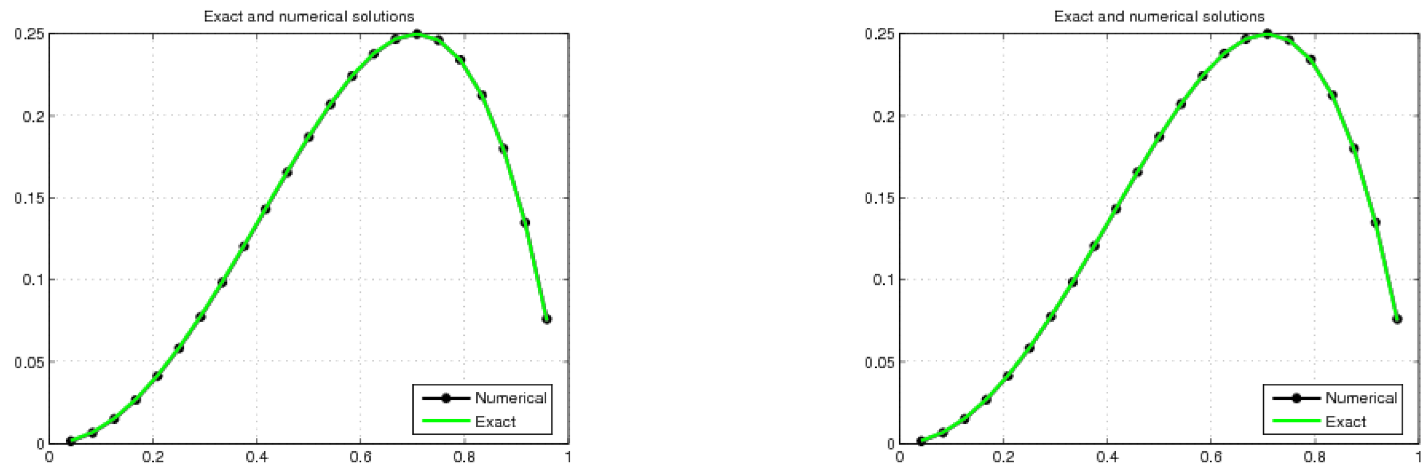
$$\frac{\partial u}{\partial t} + \frac{2}{\delta^2} \int_{x-\delta}^{x+\delta} \frac{u(x,t)-u(x',t)}{|x-x'|^{1+2s}} dx' = (x^2 - x^4) k \cos(t)$$
$$-\frac{4\delta^{-s}(\delta^2(2s-2) + (2s-4)(6x^2-1))}{(2s-4)(2s-2)} k \sin(t),$$
$$u(x,t) = (x^2 - x^4) k \sin(t), x \in [-\delta, 0] \cup [1, 1+\delta],\ t \in (0,T],$$
$$u(x,0) = 0, x \in (0,1).$$

The analytical solution is:

$$u(x,t) = (x^2 - x^4) k \sin(t),\ x \in [-\delta, 1+\delta],\ t \in [0,T].$$

Examples

For the Backward Euler method, letting $s = \frac{1}{3}$, $T = \frac{\pi}{2}, k = 1$, we have



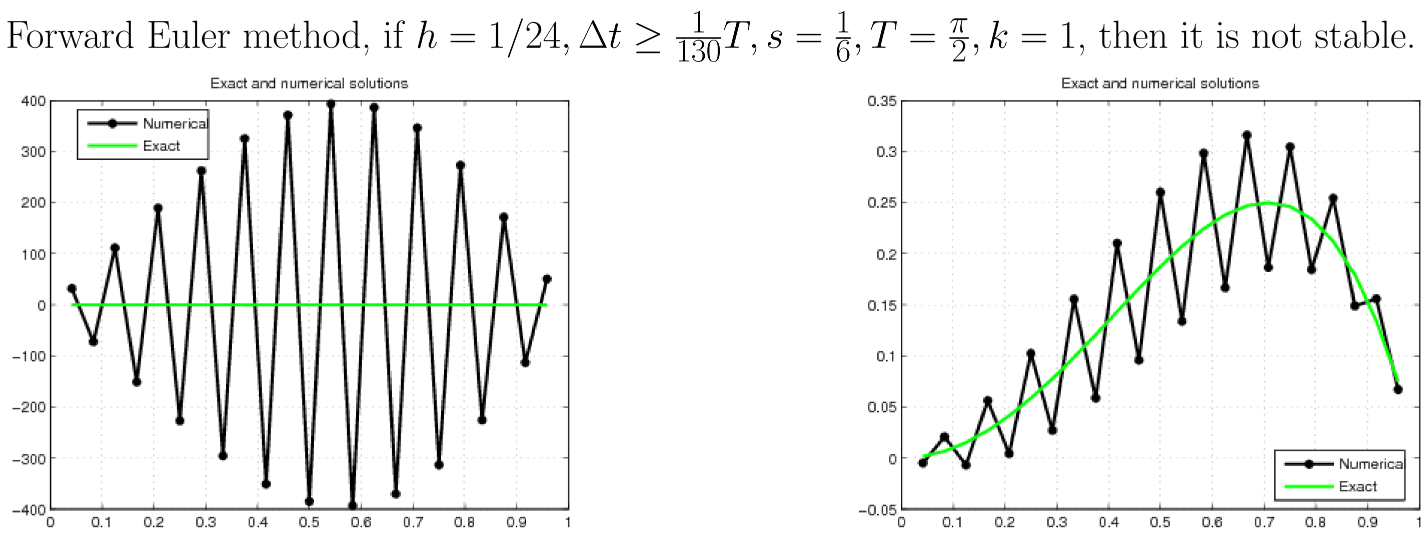
(a) $\Delta t = \frac{1}{12}T$ (b) $\Delta t = \frac{1}{24}T$

Figure 1: $h = 1/24$, for fixed $\delta = 0.2$, at $t = \pi/2$.

For $s = 9/20,\ T = 1,\ k = 10,\ \delta = 0.1$ the convergence rate is:

h	Δt	L_2 norm error	rate
1/8	$1/8^2$	0.042808	-
1/16	$1/16^2$	0.010785	1.9889
1/24	$1/24^2$	0.0047246	2.0356
1/32	$1/32^2$	0.0026076	2.0660
1/40	$1/40^2$	0.0016471	2.0588
1/48	$1/48^2$	0.0011349	2.0429

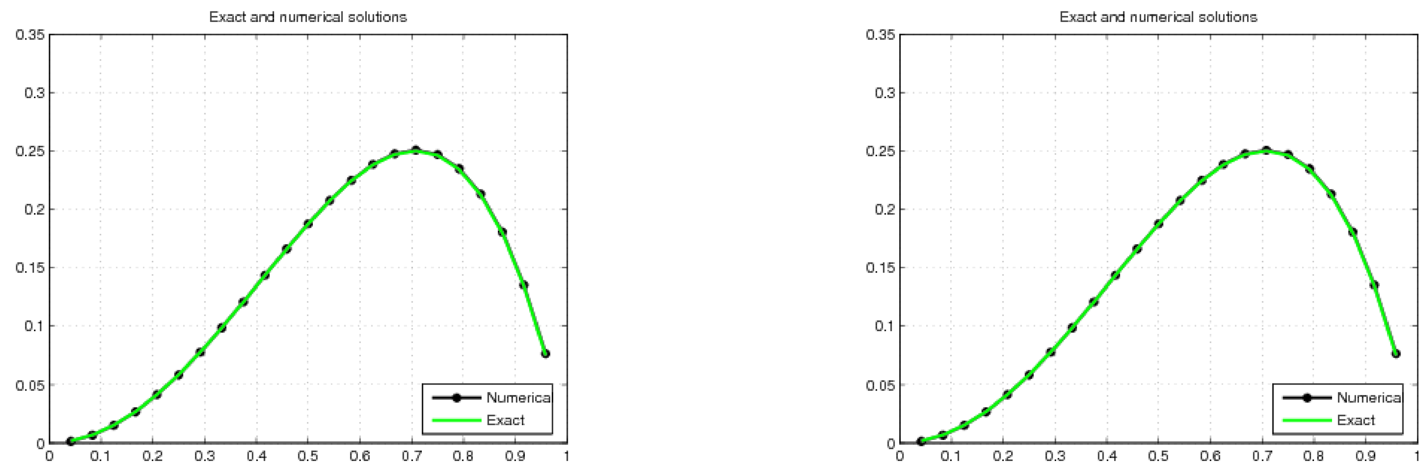
For the Forward Euler method, if $h = 1/24, \Delta t \ge \frac{1}{130}T, s = \frac{1}{6}, T = \frac{\pi}{2}, k = 1$, then it is not stable.



(a) $\Delta t = \frac{1}{125}T$ (b) $\Delta t = \frac{1}{130}T$

Figure 2: $h = 1/24$, for fixed $\delta = 0.5$, at $t = \pi/2$.

If $\Delta t \le \frac{1}{135}T$, then the method is stable.



(a) $\Delta t = \frac{1}{135}T$ (b) $\Delta t = \frac{1}{150}T$

Figure 3: $h = 1/24$, for fixed $\delta = 0.5$, at $t = \pi/2$.

For $s = 1/3,\ T = 1,\ k = 1,\ \delta = 0.4$, we have the convergence rate:

h	Δt	L_2 norm error	rate
1/24	$1/24^2$	0.00030115	-
1/32	$1/32^2$	0.00016252	2.1441
1/40	$1/40^2$	0.00010128	2.1193
1/48	$1/48^2$	6.9065e-05	2.0998
1/56	$1/56^2$	5.0091e-05	2.0837

References

[1] Qiang Du,Max Gunzburger,R. B. Lehoucq and Kun Zhou, Analysis and Approximation of Nonlocal Diffusion Problems with Volume Constraints, *SIAM Review* Vol. 54, No. 4, pp. 667-696, 2012.

[2] Endre Süli, *Finite Element Methods for Partial Differential Equations,* Oxford University Computing Laboratory, 2002.

[3] Vincent J. Ervin, Norbert Heuer, and John Paul Roop, Numerical Approximation of a Time Dependent, Non-linear,Fractional Order Diffusion Equation, *SIAM J. Math. Anal* Vol. 45, No. 2, pp. 572-591, 2007.