A Finite Element Method for the Advection-Diffusion Equation

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Abstract

Recent increases in computing power have changed the way we are able to solve PDEs. One of the more popular ways to computationally solve differential equations is the finite element method (FEM). The objective of this project is to solve the 1-D advection-diffusion equation using this method in C++. For flows that are diffusion-dominated, the standard FEM approach can be used. However, advection-dominated flows result in physically unrealistic solutions. Therefore, we have to consider methods which preserve positivity, such as flux corrected transport. We will present numerical results for diffusion-dominated and purely advection-driven flow.

Advection-Diffusion Equation

The advection-diffusion equation is given by

$$\begin{aligned} u_t(x,t) - \nu u_{xx}(x,t) + a u_x(x,t) &= f(x,t) \\ u(x,0) &= u_0(x) \end{aligned} \tag{1}$$

where ν and a are the diffusion and advection constants, respectively, with Dirichlet boundary conditions.

For the fully discrete FEM we use a weak formulation where a backward Euler approximation is used in time.

$$\int_0^1 \frac{u(x,t) - u(x,t-\Delta t)}{\Delta t} v\,dx + \nu \int_0^1 u_x v_x\,dx + a\int_0^1 u_x v\,dx = \int_0^1 f v\,dx \tag{2}$$

Using continuous piecewise linear polynomials to discretize our weak problem, we first show that we can accurately approximate the solution when the flow is diffusion-dominated.

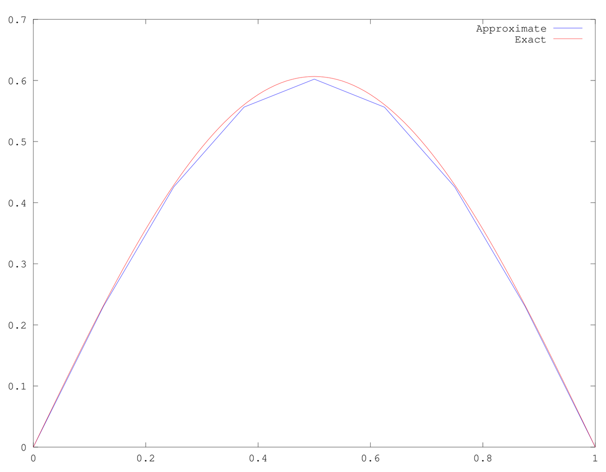

Figure 1: Exact solution and approximate solution of a diffusion-dominated flow with $t = 0.5$, $\Delta x = \frac{1}{8}$, $a = 0.002$, and $\nu = 1$

The following is a table of L_2 errors generated from Fig 1.

Δx	L_2-error	rate
$\frac{1}{4}$	0.0305977	-
$\frac{1}{8}$	0.00855629	1.83837
$\frac{1}{16}$	0.00225692	1.92263
$\frac{1}{32}$	0.000589784	1.9361

Using this standard FEM scheme to solve this equation results in significant inaccuracies if the flow is advection-dominated, as displayed in the figure below.

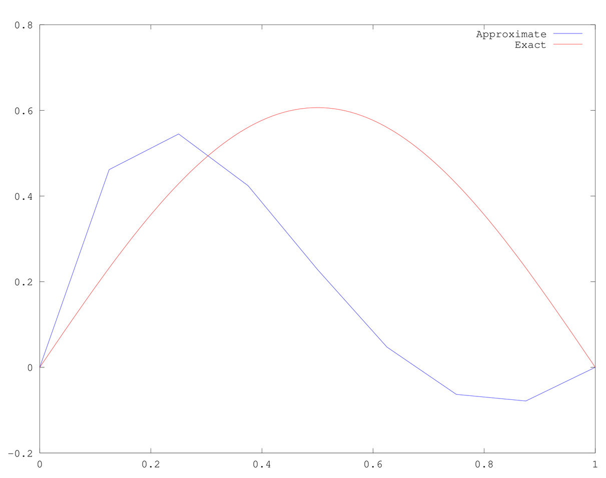

Figure 2: Exact solution and approximate solution of an advection-dominated flow with $t = 0.5$, $\Delta x = \frac{1}{8}$, $a = 1$, and $\nu = 0.2$

FEM Lax-Wendroff

To achieve a better approximation to the advection-diffusion equation, we first examine only the advection equation and try to obtain a scheme which accurately approximates its solution. Then we will incorporate this into our advection-diffusion scheme.

We use the 2nd order Lax-Wendroff scheme to approximate u_t so we can arrive at the weak form of the FEM Lax-Wendroff (FEM-LW) approximation. Let $u_{tt}(x,t) = -a^2 u_{xx}(x,t)$ to get

$$\int_0^1 \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} v\,dx + a\int_0^1 u_x v\,dx + \frac{\Delta t}{2} a^2 \int_0^1 u_x v_x\,dx = 0 \tag{3}$$

The figure below demonstrates the numerical overshoots and undershoots in the FEM-LW approximation.

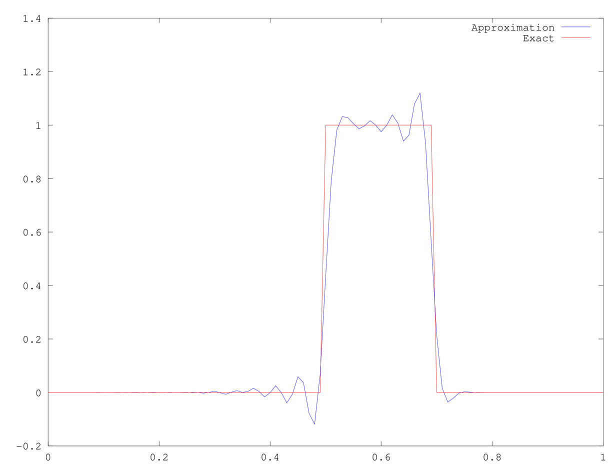

Figure 3: Exact solution and FEM-LW approximate solution of a square wave with $t = 0.5$ and $\Delta x = 0.01$

FEM Backward Euler

Similarly, we approximated u_t in the advection equation (3) above using the Backward Euler method. Thus, the finite element weak form is

$$\int_0^1 \frac{u(x,t) - u(x,t-\Delta t)}{\Delta t} v\,dx + a\int_0^1 u_x v\,dx = 0 \tag{4}$$

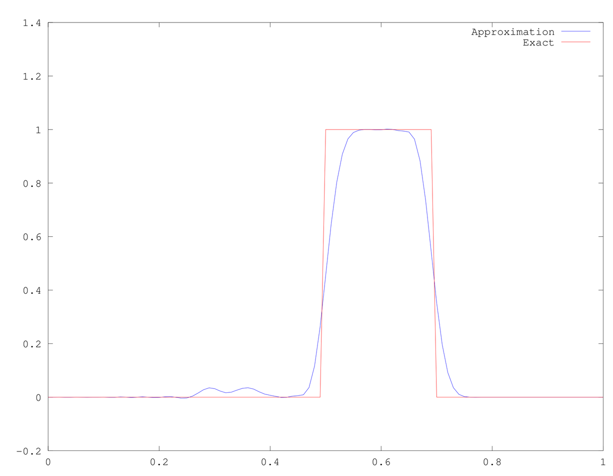

Figure 4: Exact solution and FEM-BE approximate solution of a square wave with $t = 0.5$ and $\Delta x = 0.01$

References

[1] D. Kuzmin and S. Turek, Flux correction tools for finite elements. *J. Comput. Phys.* 175 (2002) 525-558.

FEM-FCT

FEM-FCT, or FEM Flux-Corrected Transport is a method to correct the overshoots and undershoots apparent in other methods such as FEM-LW and FEM-BE. To do this, a lower-order scheme that preserves positivity will be "corrected" to achieve higher order accuracy. FEM-LW (3), when implemented, results in a matrix system to solve which can be written as

$$M\Delta u^H = \Delta t K^H u^n - \frac{(\Delta t)^2}{2} a^2 S u^n \tag{5}$$

where u^n is the solution at the previous time step, and $\Delta u^H = u^{n+1} - u^n$. We consider the lower order scheme

$$M^L \Delta u^L = \Delta t K^L u^n$$

where M^L is the lumped mass matrix, and K^L is a modification to K^H such that all negative off-diagonal entries are eliminated.

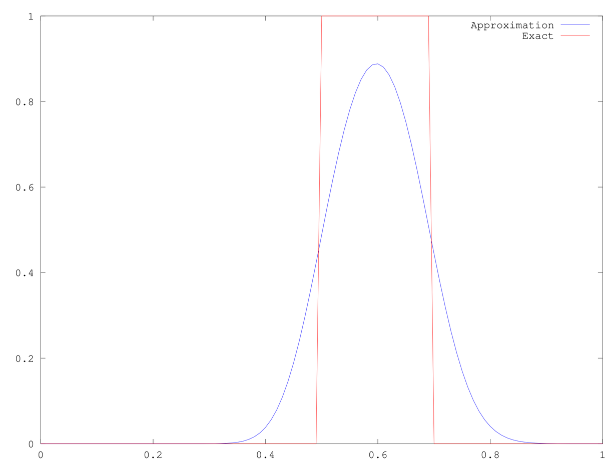

Figure 5: Exact solution and lower order approximate solution of a square wave with $t = 0.5$ and $\Delta x = 0.01$

To recover accuracy lost by the dampening effect of this lower order solution, we add back the higher order terms.

$$\begin{aligned} M\Delta u^H = {} & \Delta t K^L u^n - (M - M^L)\Delta u^H \\ & + \Delta t (K^H - K^L) u^n - \frac{(\Delta t)^2}{2} S u^n \end{aligned} \tag{6}$$

which can be written as a difference in fluxes across elements

$$(\Delta u^H)_i = (\Delta u^L)_i + \frac{1}{M_i^L} \sum_{j \neq i} a_{ij} f_{ij}$$

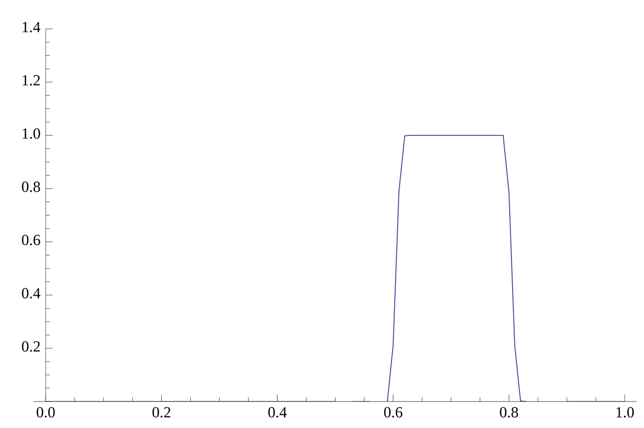

Figure 6: FCT approximate solution of a square wave with $t = 0.5$ and $\Delta x = 0.01$

Future Work

To finish, we will implement the diffusion term in the FEM-FCT routine.