Reduced Order Modeling Applied to Non-Local Problems

David Witman

Advisors: Max Gunzburger & Janet Peterson

Department of Scientific Computing, Florida State University, Tallahassee, Florida

Introduction

Nonlocal theories refer to a set of concepts and methods that can be used in lieu or in combination of currently existing techniques used to solve differential equations. Peridynamics is a class of methods contained within these nonlocal theories that was developed in order to accurately model discontinuities in certain applications like fracture dynamics. These nonlocal methods rely on what is commonly referred to as a horizon, or radius in which surrounding particles interact with one another.

Reduced Order Modeling (ROM) is a widely used class of methods used to reduce the computational cost of solving differential equations using standard techniques like the Finite Element Method (FEM) among others. The ROM method that we will use in this research is called the Proper Orthogonal Decomposition (POD) and makes use of the Singular Value Decomposition (SVD) to develop an intelligent set of reduced basis functions.

This research will demonstrate the viability of using ROM coupled with a nonlocal approach to solving a one dimensional time dependent nonlocal equation. In order to construct our reduced order nonlocal solution we will follow the provided steps:

1. Solve the differential equation a large number of times while varying the parameter inputs to understand how the solution behaves with respect to a given set of inputs. This is called collecting snapshots.
2. Use SVD on the snapshot set to form a reduced set of basis functions that intelligently captures the major features and trends of how the differential equation responds to different input parameters.
3. Construct the reduced order solution to the differential equation using as few reduced basis functions as it takes to accurately represent the solution to the differential equation.

We will attempt to follow these general steps in order to construct a reduced order solution to our one dimensional time dependent nonlocal equation.

Solving the Non-Local problem

The first item of business in constructing our reduced order nonlocal solution is solving the nonlocal equation a number of times with varying parameter inputs to see how the solution responds. But before this, we must define the nonlocal equation that we wish to solve.

S. A. Silling first introduced the concept of defining a horizon where only particles within some radius are allowed to interact, or "see" one another. By applying his theories and methodology, it can be shown that one can form the following integro-differential equation:

$$\rho\ddot{\mathbf{u}}(\mathbf{x},t) = \int_{H_x} c\frac{(\mathbf{x}'-\mathbf{x})\otimes(\mathbf{x}'-\mathbf{x})}{|\mathbf{x}'-\mathbf{x}|^3}(\mathbf{u}(\mathbf{x}',t)-\mathbf{u}(\mathbf{x},t))dV_x + \mathbf{f}(\mathbf{x},t)$$

where ρ is the mass density in the reference configuration, $\boldsymbol{x}$ and $\boldsymbol{x}'$ are the different locations of the two particles within our reference frame, H_x is the reference frame itself, c represents a constant accounting for material properties and spatial dimensions and $\mathbf{f}(\boldsymbol{x},t)$ is a given body force density. Now because we wish to make this a time dependent problem we must add in a temporal derivative, making our differential equation take the form:

$$\begin{cases} a_1\dfrac{du(x,t)}{dt} + a_2\dfrac{1}{\delta^2}\displaystyle\int_{x-\delta}^{x+\delta}\frac{u(x,t)-u(x',t)}{|x-x'|}dx' = f(x,t), & x\in\Omega \\ u(x) = g(x) & x\in\Gamma \end{cases} \quad \textbf{[1]}$$

where a_1 and a_2 are problem specific constants and δ is the horizon length.

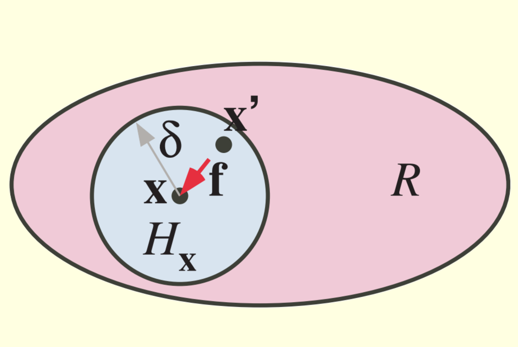


Figure 1. An illustration of how particles in the reference frame or within the horizon itself interact but there are no interactions outside of this frame

Non-Local cont.

In equation [1] we notice that the domains Ω and Γ were defined without really being explained. One thing that we must recognize with nonlocal problems is that with the inclusion of the horizon, we have integrals that extend past the ends of our domain. So assuming that γ and β represent the left and right hand sides of our domain respectively, we have the regions:

$$\begin{aligned} \Omega &= (\gamma,\beta), \\ \Omega' &= (\gamma-\delta,\beta+\delta) \\ \Gamma &= \bar{\Omega}'\backslash\Omega = [\gamma-\delta,\gamma]\cup[\beta+\delta,\beta] \end{aligned} \quad \textbf{[2]}$$

Using this information, we see that Ω represents the actual domain we would like to define our solution on and Γ is the region extending past the domain in an attempt to enforce "Dirichlet" boundary conditions. The value of these functions is assumed to be known using the function $g(x)$.

After defining our equation, the next step is to apply the Galerkin finite element approach to discretize this problem to solve for the solution to the differential equation as a linear combination of a set of unknowns and basis functions. Using this technique and approximating the temporal derivative with the backwards difference method, we can find that the weak form of our solution becomes:

$$\frac{a_1}{\Delta t}\sum C_j^k \int_{\Omega'}\phi_j(x)\phi_i(x)dx + \frac{a_2}{\delta^2}\sum C_j^k\int_{\Omega'}\phi_i(x)\int_{\Omega'}\frac{\phi_j(x)-\phi_j(x')}{|x-x'|}dx'dx = \int_{\Omega'}f(x,t)\phi_i(x)dx + \frac{a_1}{\Delta t}\sum C_j^{k-1}\int_{\Omega'}\phi_j(x)\phi_i(x)dx$$

Plus some known boundary condition data. Unfortunately if we simply use this as the weak formulation to construct our linear system and solve for our unknown solution vector, we will encounter some stability problems. It can be shown that when this approach is approximated with a quadrature rule, it creates an unstable system due to the setup of the second term. Fortunately, we can modify the second term in such a way that the system is stable. Modifying this weak form leaves us with the fully discrete weak formulation:

$$\frac{a_1}{\Delta t}\sum C_j^k \int_{\Omega'}\phi_j(x)\phi_i(x)dx + \frac{a_2}{2\delta^2}\sum C_j^k\int_{\Omega'}\int_{\Omega'}\frac{\phi_j(x)-\phi_j(x')}{|x-x'|}(\phi_i(x)-\phi_i(x'))dx'dx = \int_{\Omega'}f(x,t)\phi_i(x)dx + \frac{a_1}{\Delta t}\sum C_j^{k-1}\int_{\Omega'}\phi_j(x)\phi_i(x)dx \quad \textbf{[3]}$$

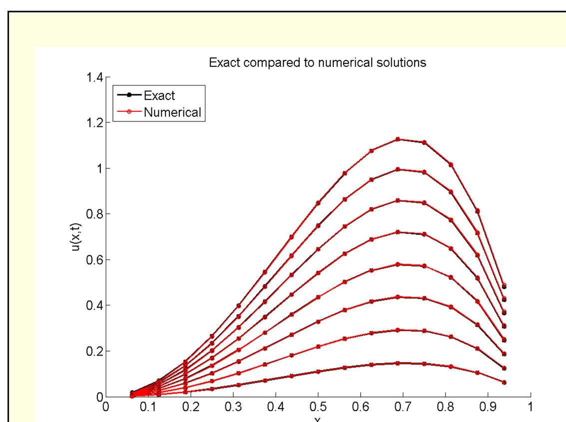


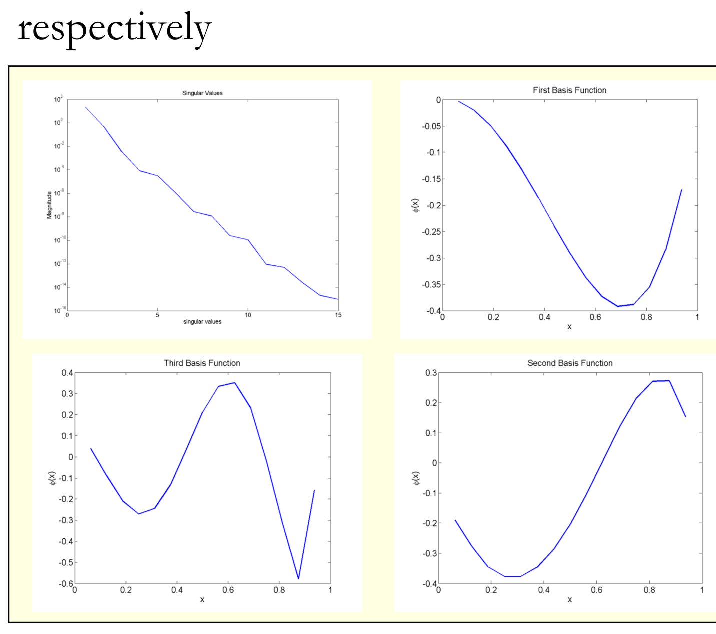
Figure 2. Solution to differential equation at various time steps using method of manufactured solutions where: $u_{exact} = 10(x^2(1-x^2))\sin(t)$

Figure [2] shows the exact and computed numerical solutions to the fully discrete weak formulation when we use piecewise linear basis functions with our horizon set to $\delta = 0.125$. Using the exact solution and the computed solutions at a number of discretization's we can calculate the rates of convergence as the resolution is increased.

h	Euclidean distance Error	Rate
0.25	2.508895e-002	
0.125	4.932741e-003	2.439799
0.0625	1.025740e-003	2.093973
0.03125	2.204433e-004	1.921912

Proper Orthogonal Decomposition

Now that we have a set of solutions computed for a given set of input parameters the next step is to compute the reduced basis function. Because we stored our solutions in a column-wise format, we extract the column space basis vectors from the U matrix in the decomposition $A = USV^T$. Figures [3]-[6] show the singular values and the first three basis functions respectively



Figures [3]-[6]. Top left moving clockwise: singular values, first three reduced basis functions

Reduced Order Modeling

The final piece of the puzzle is now to form our reduced order solution to the nonlocal equation using our reduced basis functions. From figures [4]-[6] we can see that our reduced basis functions are no longer compactly supported like our piecewise linear basis functions. In fact, the reduced basis functions are defined over the entire domain, Ω. This means that our linear system will no longer be banded with respect to the choice of basis function and the horizon size. In general, our resultant linear system will now be small and dense. The advantage comes in to play with the fact that our reduced basis functions have knowledge of how the solution behaves with respect to input parameters. Thus we hope that the number of basis functions it takes to represent the solution is small so that it is more efficient to solve a small dense linear system than a large banded system.

To start off, we will define our reduced order solution as a linear combination of a set of unknowns and the reduced basis functions which are in turn, a linear combination of the basis vector values and the piecewise linear basis functions:

$$u_{ROM}(x,t_k) = \sum\sigma_j^k\Psi_j(x) \text{ where } \Psi_j(x) = \sum C_j^k\phi_i(x)$$

where σ^k is the set of unknown values that we will solve for and Ψ are the reduced basis functions themselves.

Now that we have defined our reduced basis functions, we simply have to substitute in the new definition of said basis functions into our fully discrete weak formulation. We recall that when formulating the weak form of the nonlocal equation, we were required to make some modifications in order to account for stability in the linear system. Fortunately all the assumptions we made, still hold for the reduced basis case, so we can write the second term in our reduced order weak formulation as:

$$A_{i,j} = \frac{a_2}{2\delta^2}\sum_{x_l=\alpha}^{x_l=\beta} w_l \sum_{x_m=x_l-\delta}^{x_m=x_l+\delta} w_m \frac{\sum C_j^k\phi_j(x_l) - \sum C_j^k\phi_j(x_m)}{|x_l-x_m|} \left(\sum C_i^k\phi_i(x_l) - \sum C_i^k\phi_i(x_m)\right) + \ldots$$

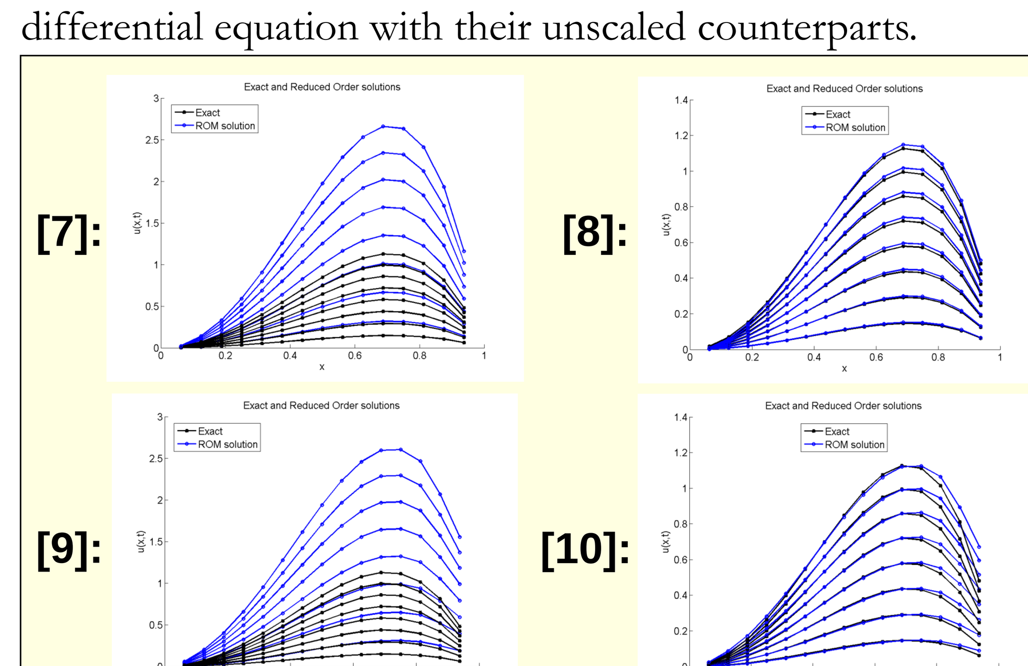
The remaining terms are straightforward to implement and has been done numerous times in many works associated with ROM. To complete our new weak formulation we have:

$$A_{i,j} = \frac{a_1}{\Delta t}\int_\alpha^\beta \sum C_j^k\phi_j(x)\sum C_i^k\phi_i(x)dx + \ldots$$

and:

$$b_i = \int_\alpha^\beta f(x,t)\sum C_i^k\phi_i(x)dx + \frac{a_1}{\Delta t}\sum\alpha_j^{k-1}\int_\alpha^\beta\sum C_j^k\phi_j(x)\sum C_i^k\phi_i(x)dx$$

Using this new weak formulation based off of the reduced basis functions, we can setup and solve a small dense linear system for the unknown coefficients, σ^k. Figures [7]-[10] illustrate some sample reduced order solutions to our differential equation with their unscaled counterparts.



Figures [7],[9]: Unscaled ROM solutions (1,5 basis functions included)
Figures [8],[10]: Scaled solutions (1,5 basis functions included)

Bibliography and Acknowledgement

1. Xi Chen and Max Gunzburger. Continuous and discontinuous finite element methods for a peridynamics model of mechanics. *Computer Methods in Applied Mechanics and Engineering*, 200(9):1237–1250, 2011.
2. Stewart A Silling. Reformulation of elasticity theory for discontinuities and long-range forces. *Journal of the Mechanics and Physics of Solids*, 48(1):175-209, 2000.

Finally I would like to thank Max Gunzburger for support of my research this past semester.