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# Solution of the Navier Stokes Equation with a Colored Noise Forcing Term

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Abstract: We pose a version of the time-dependent incompressible Navier-Stokes equations with a stochastic forcing term. The finite element method is used to discretize the variational form of the problem. The stochastic forcing term is represented by a covariance function whose eigenvalues are employed in a truncated Karhunen-Loeve expansion. Finite element computations are applied to problems with both Gaussian and exponential covariance functions, and the appropriate rate of convergence is observed.

## Introduction

Formally, the stochastic incompressible Navier Stokes equations with Newtonian constitutive relationship may be written as:

$$\begin{aligned} \mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f}(t,x,\omega) &\text{ in } (0,T)\times\mathcal{D}\times\Omega,\\ \nabla\cdot\mathbf{u} = 0 &\text{ in } (0,T)\times\mathcal{D}\times\Omega,\\ \mathbf{u} = \mathbf{g}(t,x) &\text{ on } (0,T)\times\partial\mathcal{D},\\ \mathbf{u} = \mathbf{u}_0(x) &\text{ on } \mathcal{D}\times\{0\}. \end{aligned}$$

A colored noise function $\mathbf{f}(t,x,\omega)$ in space has an associated semidefinite covariance function $C(x,y)$. Thus, the relationship between two values of the forcing term can be measured by

$$< \mathbf{f}(t,x,\omega), \mathbf{f}(s,y,\omega) >= \delta(t-s)C(x,y)$$

whe $\delta(t)$ is the usual Dirac delta function.

The corresponding stochastic variational formulation:

$$\begin{aligned} \int_{\mathcal{D}} \mathbb{E}[\partial_t\mathbf{u}\cdot\mathbf{v}]dx + \nu\int_{\mathcal{D}}\mathbb{E}[\nabla\mathbf{u}:\nabla\mathbf{v}]dx + \int_{\mathcal{D}}\mathbb{E}[(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}]dx - \int_{\mathcal{D}}\mathbb{E}[p\nabla\cdot\mathbf{v}]dx \\ = \int_{\mathcal{D}}\mathbb{E}[\mathbf{f}\cdot\mathbf{v}]dx \\ \int_{\mathcal{D}}\mathbb{E}[\psi\nabla\cdot\mathbf{u}]dx = 0 \end{aligned}$$

where $\mathbf{v}\in S$ and $\psi\in Q$.

$$\begin{aligned} \mathbf{S} &= \{\mathbf{v}\in\mathbf{H}_0^1 : \nabla\cdot\phi(\cdot,\omega) = 0, P\text{-}a.e.\}. \\ \mathbf{Q} &= \{p\in L^2(\mathbf{u}) : \int_{\mathcal{D}} p(\cdot,\omega)dx = 0, p\text{-}a.e.\} \end{aligned}$$

$\mathbf{H}_0^1 \equiv [\widetilde{H}_0^1]^d$ equipped with $||v||_{\widetilde{W}^{s,q}(\mathcal{D})} = (\mathbb{E}[||v||_{H_0^1(\mathcal{D})}])^{1/2}$.

## Monte Carlo Galerkin Finte Element Method

In this section we discribe the use of the standard Monte Carlo Galerkin finite element method to construct approximatations of each realization.

Given a number of realizations, M, and the finite element space $S_h, Q_h$ on $\mathcal{D}$. For each j=1,2,...,M, sample independent and identically distribution of the external random force $f(t,\cdot,\omega_j)$ based on realization of KL expansion. and find a corresponding approximation $\mathbf{u}_h(t,\cdot,\omega_j)\in S_h, q_h(t,\cdot,\omega)\in Q_h$.

$$\begin{aligned} \int_D \frac{\partial\mathbf{u}(\cdot,\omega_j)}{\partial t}\cdot\mathbf{v}dx + \int_D(\mathbf{u}(\cdot,\omega_j)\cdot\nabla)\mathbf{u}(\cdot,\omega_j)\cdot\mathbf{v}dx + \nu\int_D\nabla\mathbf{u}(\cdot,\omega_j):\nabla\mathbf{v}dx - \\ \int_D p(\cdot,\omega)\nabla\cdot\mathbf{v}d\Omega = \int_D\mathbf{f}(\cdot,\omega_j)\cdot\mathbf{v}dx \\ \int_D \phi\nabla\cdot\mathbf{u}(\cdot,\omega_j)dx = 0 \end{aligned}$$

where $\mathbf{v}(t,\cdot,\omega_j)\in S_h, \phi(t,\cdot,\omega_j)\in Q_h$ as-P.

By the Karhunen-Loeve Representation theorem, The colored noised right hand side $\mathbf{f}(x,\cdot,\omega):\mathcal{D}\times\Omega\to R$ with mean $\mu_f(x)$ and covariance kernal $C(x_1,x_2)$ can be represented as

$$\mathbf{f}(x,\cdot,\omega) = \mu(x) + \sum_{i=1}^{\infty}\sqrt{\lambda_i}e_i(x)\xi_i(\omega) \text{ in } L^2(\Omega)\text{-}a.e.$$

where $\xi_i$ are centered mutually uncorrelated random variables with unit variance, $\{\lambda_i, e_i\}$ are the eigenvalues and orthonormal eigenfunctions of the Fredhelm integration equation of second kind

$$\int_D C(x,y)e_j(y)dy = \lambda_j e_j(x), j = 1,2,..., \tag{1}$$

with $\mu_1 \geq \mu_2 \geq \cdots \geq 0$.

## Time discretization

Applying the backward Euler method, This leads to fully implicit method for seeking $\mathbf{u}_h$ in $n+1$-st time layer:

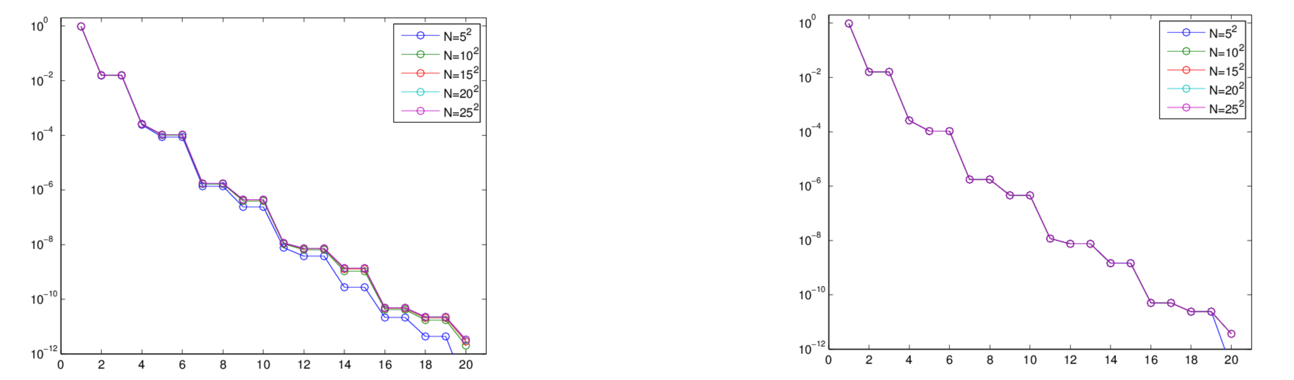
$$\begin{aligned} \frac{1}{\Delta t}\int_{\mathcal{D}}\mathbf{u}_h^{n+1}(\omega_j)\cdot\mathbf{v}_h dx + \int_{\mathcal{D}}(\mathbf{u}_h^{n+1}(\omega_j)\cdot\nabla)\mathbf{u}_h^{n+1}(\omega_j)\cdot\mathbf{v}_h dx + \nu\int_{\mathcal{D}}\nabla\mathbf{u}_h^{n+1}(\omega_j):\nabla\mathbf{v}_h dx - \\ \int_{\mathcal{D}} p_h^{n+1}(\omega_j)\nabla\cdot\mathbf{v}_h dx = \int_{\mathcal{D}}\mathbf{f}^{n+1}(\omega_j)\cdot\mathbf{v}_h dx + \frac{1}{\Delta t}\int_{\mathcal{D}}\mathbf{u}_h^n\cdot\mathbf{v}_h dx, \\ \int_{\mathcal{D}}\psi_h\nabla\cdot\mathbf{u}_h^{n+1}(\omega_j)dx = 0. \end{aligned}$$

The resulting nonlinear algebraic system is then solved by the Newton Method. In the inner iterations the umfpack solver are employed to solve the linear system.

## Guassian Colored Noise Stimulation

$$C_f(x,y) = \sigma^2 e^{-\frac{|x-y|^2}{L_c}}, x,y\in\mathcal{D}$$

After orthogonalizing $e_i$ with same eigenvalues (eg.Gram Schmidt) and normalizing eigenvectors $e_i$ with numerical integration scheme $\sum_{i=1}^N w_n e_i(x_i)e_i(x_i)$, we can find a orthonormal basis $\{e_i\}$ with quadrature weights $\{w_i\}$ which is good numerical approximation of $e_i$ of (1).



(a) Piecewise Middle points rule

(b) Guass-Legendre quadrature rule

Figure 1: the first few eigenvalues with different different quadrature points

## Examples

we consider the following two-dimensional stochastic Navier Stokes driven by color noise.

$$\begin{aligned} \mathbf{u}_t - \nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p = f(x,t) + \xi(x,t,\omega) &\text{ in } (0,T)\times\mathcal{D}\times\Omega,\\ \nabla\cdot\mathbf{u} = 0 &\text{ in } (0,T)\times\mathcal{D}\times\Omega,\\ \mathbf{u} = \mathbf{g} &\text{ on } (0,T)\times\partial\mathcal{D}\times\Omega,\\ \mathbf{u} = \mathbf{u}_0 &\text{ on } \mathcal{D}\times\{0\}\times\Omega. \end{aligned}$$

Where $\xi(x,t,\omega)$ denotes the color noise with mean zeros and gaussian variance function $C_f(x,y) = \sigma^2 e^{-\frac{|x-y|^2}{L_c}}, x,y\in D$ ,$\sigma = 1$ and $\nu = 1, L_c = 10$, and

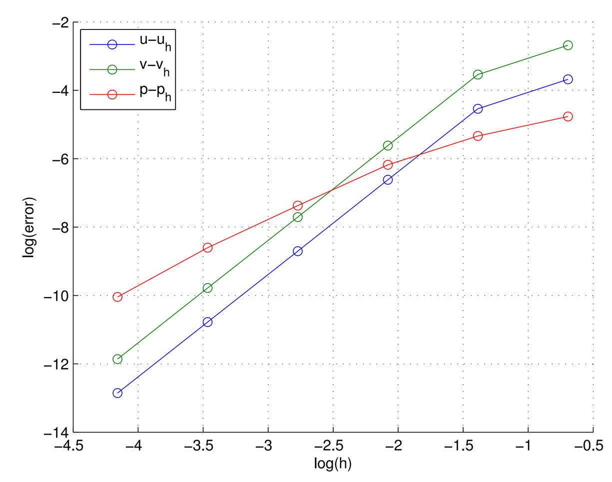
$$\mathbf{g}(t,x) = (e^{-t}\cos(2\pi y)\sin(2\pi x), -e^{-t}\cos(2\pi x)\sin(2\pi y))$$

$$\begin{aligned} f(x,t) =&(2x + \pi e^{-2t}\sin(4\pi x) - e^{-t}\cos(2\pi y)\sin(2\pi x) + 8\pi^2\nu e^{-t}\cos(2\pi y)\sin(2\pi x), \\ &2y + \pi e^{-2t}\sin(4\pi y) + e^{-t}\cos(2\pi x)\sin(2\pi y) - 8\pi^2\nu e^{-t}\cos(2\pi x)\sin(2\pi y)) \end{aligned}$$

$$\mathbf{u}_0 = (\cos(2\pi y)\sin(2\pi x), \cos(2\pi x)\sin(2\pi y))$$

| h | $\|\|u(T)-u_h(T)\|\|$ | order | $\|\|v(T)-v_h(T)\|\|$ | order | $\|\|p(T)-p_h(T)\|\|$ | order |
|---|---|---|---|---|---|---|
| 1/2 | 2.524270e-02 | - | 2.524270e-02 | - | 1.263379e+00 | - |
| 1/4 | 1.069344e-02 | 1.239141 | 1.068352e-02 | 1.240480 | 7.144075e-01 | 0.822468 |
| 1/8 | 1.337853e-03 | 2.998734 | 1.336509e-03 | 2.998846 | 3.077256e-01 | 1.215103 |
| 1/16 | 1.656229e-04 | 3.013946 | 1.651406e-04 | 3.016702 | 9.342183e-02 | 1.719813 |
| 1/32 | 2.081968e-05 | 2.991882 | 2.076527e-05 | 2.991451 | 2.723982e-02 | 1.778043 |
| 1/64 | 2.609404e-06 | 2.996156 | 2.591168e-06 | 3.002498 | 6.451420e-03 | 2.078028 |

Table 1: the computational results for 100 simulations




## References

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