An Optimization-Based Model Coupling Method Using Slightly Overlapping Subdomains

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Abstract

In this research project, we explore the possibility of coupling two models together across a sharp interface via an optimal control methodology. We present solutions for coupled Poisson equations with varying material constants and forcing terms. For the test cases presented, a projection of the Neumann boundary condition at the interface onto the discrete interface boundaries was used as the control parameter. Through visual evidence along with preliminary numerical convergence studies, we have reason to believe that our proposed method will able to accurately solve interface coupling problems.

Introduction

Many physical systems that must be modeled require more than one model to be solved on separate subdomains of the problem region with a sharp interface condition that governs how a physical quantity is conserved or transfered across the subdomain. Some examples of the problem of interest are fluid-structure interactions, vessicle deformation in fluid flow, and Stokes-Darcy flow in Karst aquafers. Unfortunately, in many cases, the sharp-interface coupling problems is unstable and it becomes difficult to find a numerical solution of the problem for this reason. Also, a direct solution of the coupled problem can be expensive since the size of the matrix increases to accomodate for the entire discretization of the problem to be solved.

In this research, we are investigating using an optimal control based method for solving the interface coupling problem. Because of the instability of the problem described above, it is clear that an optimization problem that minimizes the difference of the solutions at the interface will also be unstable. To stabilize the interface coupling problem, we make an ϵ sized perturbation of the problem subdomains to induce a small overlap at the interface. This small overlap may also be used to tie unmatching discretizations together to avoid any voids regions that may occur. To reduce the cost of the model coupling, we solve each model iteratively until the desired optimal control parameter is achieved. It is hopeful that our method will allow us to use already existing code for each model used in the interface-coupling problem so that minimal effort would be required to program our algorithm.

Our Method

Instead of solving the overlap problem directly, we decouple the models and use an iterative method to minimize the cost functional

$$\min_{u_1,u_2,g} J^{\delta,\epsilon} = \frac{1}{2}||u_1 - u_2||^2_{L^2(\Omega_0^\epsilon)} + \frac{\delta}{2}\sum_{i=1}^{2}||\mathbf{P}_i g||^2_{L^2(\Gamma_0^{i,\epsilon})} \tag{1}$$

subject to the PDE constraints

$$-k_i\Delta u_i = f_i \quad \text{in } \Omega_i^\epsilon \tag{2}$$
$$u_i = 0 \quad \text{on } \Gamma_i^\epsilon \tag{3}$$
$$k_i\frac{\partial u_i}{\partial n_i} = (-1)^{i+1}\mathbf{P}_i g \text{ on } \Gamma_0^{i,\epsilon} \tag{4}$$

where $u_i \in H_0^1(\Omega_i^\epsilon)$, $g \in H_{00}^{1/2}(\Gamma_0)$, $\delta \in \mathbb{R}$ is the regularization parameter, and $\mathbf{P}_i : H_{00}^{1/2}(\Gamma_0) \to H_{00}^{1/2}(\Gamma_0^{\epsilon,i})$ is defined as an projection from the interface to the boundaries of the overlap region.

By taking the Fréchet derivative of the Lagrange multiplier functional with resepect to u_i, we derive the adjoint equations

$$-k_i\Delta\lambda_i = (-1)^{i+1}\chi_{\Omega_o^\epsilon}(u_1 - u_2) \text{ in } \Omega_i^\epsilon \tag{5}$$
$$\lambda_i = 0 \text{ on } \Gamma_i^\epsilon \tag{6}$$
$$k_i\frac{\partial\lambda_i}{\partial n_i} = 0 \text{ on } \Gamma_0^{i,\epsilon} \tag{7}$$

By finding $\frac{dJ^{\delta,\epsilon}}{dg}$ through the first variation on the cost functional, we are able to derive the following steepest descent algorithm to solve the optimization problem presented above.

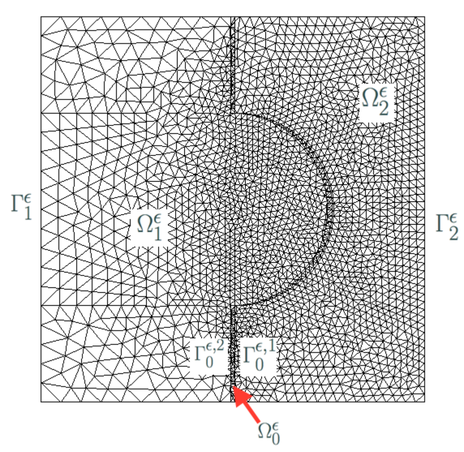

Figure 1: An example problem domain.

Algorithm 1. *Steepest Descent*

1. Map $\Omega_i \to \Omega_i^\epsilon$

2. Define the forcing functions and boundary conditions on Ω^ϵ

3. Set g^0 *to initial guess.*

4. For $n = 1, \ldots, N$

(a) Solve for u_1^n *with* $k_1\frac{\partial u_1^n}{\partial n_1} = \mathbf{P}_1 g^n$

(b) Solve for u_2^n *with* $k_2\frac{\partial u_2^n}{\partial n_2} = \mathbf{P}_2 g^n$

(c) Solve for λ_1^n *with* $k_1\frac{\partial l_1^n}{\partial n_1} = 0$

(d) Solve for λ_2^n *with* $k_2\frac{\partial l_2^n}{\partial n_2} = 0$

(e) Set $\mathbf{P}_1 g^{n+1} = \mathbf{P}_1 g^n - \frac{\alpha}{\delta}\left(\lambda_1^n - \mathbf{P}_1\lambda_2^n + \delta(\mathbf{P}_1 g^n + \mathbf{P}_2 g^n)\right)$

(f) Set $\mathbf{P}_2 g^{n+1} = \mathbf{P}_2 g^n - \frac{\alpha}{\delta}\left(\mathbf{P}_2\lambda_1^n - \lambda_2^n + \delta(\mathbf{P}_1 g^n + \mathbf{P}_2 g^n)\right)$

5. Map $\Omega_i^\epsilon \to \Omega_i$

where α is defined as the step size. In this research, we use the finite element method to solve the constraint and adjoint equations.

Numerical Convergence

In this section, we present numerical error convergence for our method. The exact solution of the test problem used is $u = \sin(2\pi x)\cos(\pi(y - 1/2))$ in $\Omega_1 = [-1, 0] \times [0, 1]$ and $u = 2\cos(\pi(x - 1/2))\cos(\pi(y - 1/2))$ in $\Omega_2 = [0, 1] \times [0, 1]$. It is found that there is an $\mathcal{O}(\epsilon)$ error that is incurred by an ϵ overlap at the interface. By setting $\epsilon = h^2$, we achieve first order convergence in $H^1(\Omega)$ and second order convergence in $L^2(\Omega)$. This error convergence is typical for piecewise linear finite element approximations.

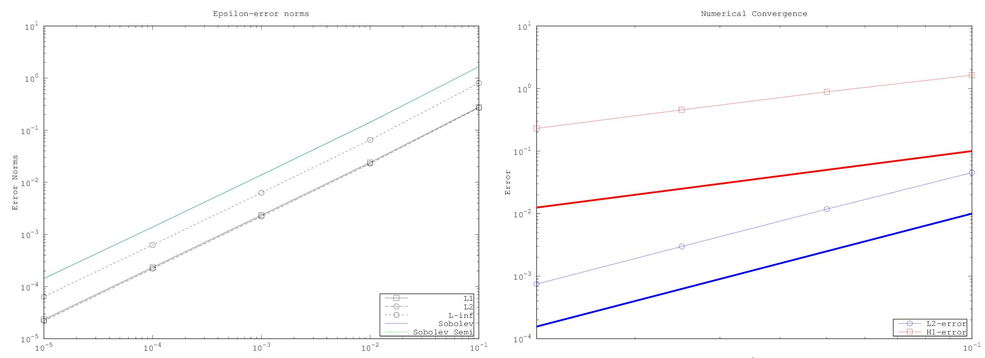

Figure 2: Log-scale error plot. **Left**: $\mathcal{O}(\epsilon)$ error. **Right**: $\mathcal{O}(\epsilon)$ error, with $\epsilon = h^2$. Red data points: $||u - u^\epsilon||_{H^1(\Omega)}$. Blue data points: $||u - u^\epsilon||_{L^2(\Omega)}$. The thick red line is a reference line with slope 2. The thick blue line is a reference line with slope 1.

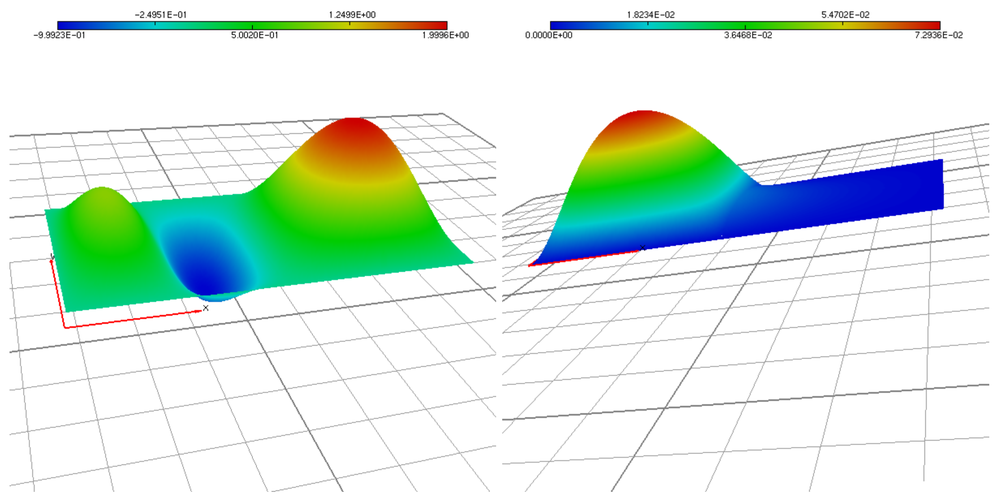

Figure 3: **Left**: Solution of the test problem obtained using interface overlap. **Right** Test problem where $f_1 = \cos(xy)$ and $f_2 = 0$ with $k_1 = 1$ and $k_2 = 10$.

Forthcoming Research

The research presented on this poster is in its preliminary stages. In the future, we will explore more complicated overlapping regions as well as modifying the cost functional. Once a robust and effective method has been established, rigorous mathematical analysis of the method will be required.

References

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