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Optimal control for networked moments

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Abstract

We study the optimal control of the mean and variance of the network state vector. We develop an algorithm that uses projected gradient descent to optimize the control input placement, subject to constraints on the state that must be achieved at a given time threshold; seeking an input placement which moves the moment at minimum cost. First, we solve the state-selection problem for a number of variants of the first and second moment, and find solutions related to the eigenvalues of the systems' Gramian matrices. We then nest this state selection into projected gradient descent to design an optimal input augmentation.

Introduction

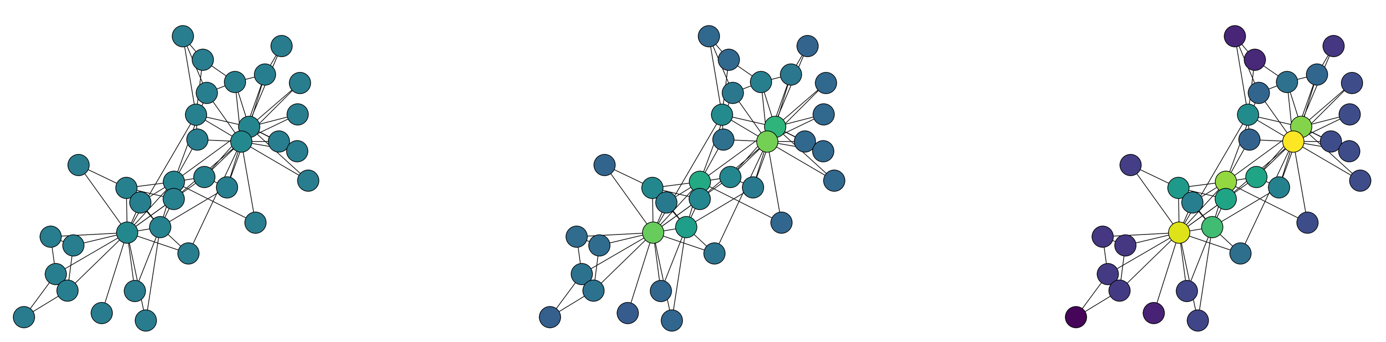
Finding a set of leaders, drivers, or actuators from which the dynamics of the system can be controlled most efficiently is an important problem, and one which has recently received an influx of attention [6, 3, 4, 8, 5]. **We approach the problem from a new angle; rather than taking the desired final state as given, we search first for the *state* that satisfies the goal at minimum cost. Then, we examine the placement of inputs to optimally adjust the output measure.**

Previous literature on input placement for efficient target control of network systems [2, 5], has focused on fully controlling a subset of nodes, rather than sufficient statistics of the state. Much literature on network control input placement has also focused on driver node selection, as opposed to augmentation [8, 3]. A notable exception is [6], which develops the Projected Gradient Method (PGM) by restricting the columns of the control input matrix, representing its schematic of outgoing connections, to be embedded on a sphere surface rather than restricted to be drawn from the canonical basis.

We frame these problems as optimal control for networks with state distribution goals. These goals come in the form of statistical features of a distribution. Specifically, the mean and the second moment or variance. With mean constraints, the problem is straightforward and can be solved using standard methods. The variance or second moment problem can be solved by nesting convex programming in the general case, and allows for a closed-form bound on the energy cost derived from the spectrum of the controllability Gramian.

Main Objectives

1. Frame influence, regulation, and discord as optimal control problems of nonlinear network state metrics, particularly the first and second moment of the network state
2. Extend target controllability and driver node selection to augmentation with metric constraint goals
3. Understand how network structure impacts control node placement in real-world networks



(a) $t^* = 0.015$ (b) $t^* = 0.150$ (c) $t^* = 1.500$

Figure 1: Evolution of flux centrality on the Zachary's karate club network

Theory

We focus on the case of finite-time control of a canonical linear time-invariant system with the following dynamics:

$$\dot{x} = Ax + Bu, \tag{1}$$

where x is a $n \times 1$ column vector containing the states of n variables, $\dot{x} \triangleq \frac{\partial x}{\partial t}$, the $n \times n$ matrix A describes the behavior of the autonomous system, u is a $m \times 1$ vector containing control input values and B is a $n \times m$ matrix which describes the nodes to which input from u is sent. The final state of this system with no input is given by $e^{t^* A}x_0$. For convenience, we will use $z \triangleq e^{t^* A}x_0$ to denote the final state of the autonomous dynamics.

We use the standard method of Lagrange multipliers to derive a minimum-energy final state for a system subject to a constraint on its central tendency through the average value of nodes at evaluation time t^*. This is an instance of the constrained eigenvalue problem of [1]. In this case, constraints take the form:

$$\frac{1}{n}\sum_{i=1}^{n} x_i \geq \eta \tag{2}$$

Under the optimal input signals, the minimum energy required to control the system from initial state x_0 to final state $x \in \text{span}(C)$ by the time $t = t^*$ (beginning, without loss of generality, at time $t = 0$) is known [7] to have the following closed-form solution

$$\mathcal{E}_{t^*}(x) = (z - x)^\top W^\dagger (z - x), \tag{3}$$

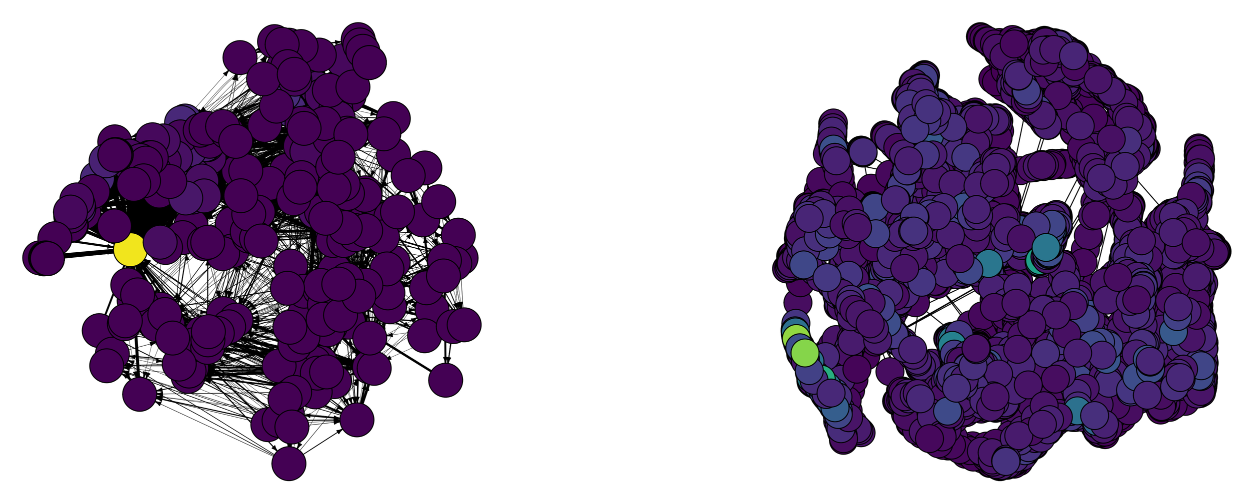
where $W^\dagger$ is the Moore-Penrose pseudoinverse of the following matrix called the *reachability Gramian*

$$W(B; A, t^*) = \int_0^{t^*} e^{tA}BB^\top e^{tA^\top} dt. \tag{4}$$

For convenience, we will define the scalar-valued function $\kappa : \mathbb{R}^{n \times m} \times \mathbb{R}^n \to \mathbb{R}$ as

$$\kappa(B; \gamma) \triangleq \gamma^\top W(B)\gamma$$

and $\alpha(\gamma) \triangleq \gamma^\top z - n\eta$ as the autonomous violation of the constraint.



(a) Neurons of the nematode *C. Elegans* (b) Western U.S. power grid

Figure 2: Structure of sample networks and flux on the time horizon $t^* = 1$

Results

Theorem 1. *The optimal state which can satisfy (2) at the lowest possible energy cost for a given control input matrix B and linear output γ is given by*

$$x_a^* = z - \frac{\alpha(\gamma)}{\kappa(B, \gamma)} W\gamma. \tag{5}$$

In this simplest case, we have a closed form expression for the energy cost associated with meeting the mean constraint given an input schematic B. To solve this problem, we will use matrix calculus to derive an analytic solution to the OMAP for control of a linear combination of network states, such as the average, and its associated energy cost.

Theorem 2. *A system with a single linear observer (such as the average state) can always be output controlled from $m \geq 1$ of controllers.*

From here, it is natural to consider the problem of controlling higher-order moments of the network state distribution. A constraint on the variance or spread of network states around some point $d \in \mathbb{R}^n$ could be written in the following form:

$$\|Ox - d\|^2 = \eta \tag{6}$$

Taking $O = I$ and $d = z$ and have the following result for state-selection:

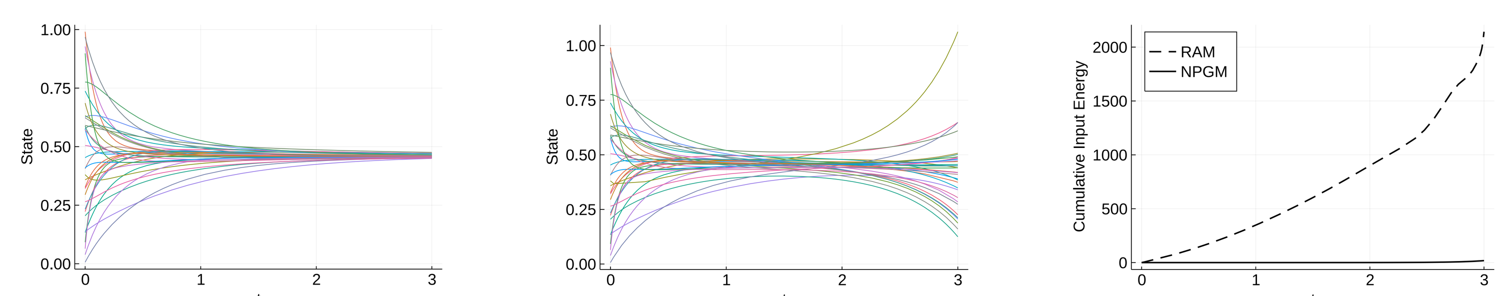
Theorem 3. *The optimal state which meets the constraint (6) at minimum energy cost under the controller schematic B is given by $x^* = z + \omega$, where ω is the leading eigenvector of $W(B)$, normalized to length $\sqrt{\eta}$.*

For the second moment, we no longer have a guarantee that a single input is sufficient.

Theorem 4. *The second-order "repulsion" moment statistic $(x - d)^\top (x - d)$ can be controlled to by a single controller to any threshold η, provided that*

$$\eta \geq \|d\|_2^2 - \|d\|_\infty^2 \tag{7}$$

To solve the OMAP for the variance control problem, we can nest the objective inside a projected-gradient method (PGM) algorithm to find a locally optimal solution.



(a) $t^* = 0.015$ (b) $t^* = 0.150$ (c) $t^* = 1.500$

Figure 3: Evolution of flux centrality on the Zachary's karate club network

Conclusions

- Control theory provides a window into the structure of complex networks
- Neural networks contain a small number of highly central nodes, relative to man-made networks
- Centrality becomes more important to average controllability as the timescale extends
- Inputs can be designed for efficient variance control using the projected gradient method

References

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